The Speed of Information Propagation in Large Wireless Networks

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***Abstract*—This paper investigates the speed limit of information propagation in large wireless networks, which provides fundamental understanding of the fastest information transportation and delivery that a wireless network is able to accommodate. We show that there is a unified speed upper bound for broadcast and unicast communications in large wireless networks. When network connectivity and successful packet delivery are considered, this speed upper bound is a function of node density. As this bound is unreachable with finite node density, we also quantify the gap between the actually achieved speed and the desired upper bound, which converges to zero exponentially as the node density increases to infinity.**

I. Introduction

Since the seminal work by Gupta and Kumar [1] on the capacity of wireless networks, there has been intensive research on understanding the fundamental properties and performance limits of wireless networks. Most work has focused on the traffic throughput, the important metric to gauge the load accommodation capability of a network. The bounds on throughput capacity have been derived under various assumptions [2]–[7], different techniques have been studied to improve these bounds [8]–[10], and algorithms have been proposed to achieve the close-to-bound throughput [11], [12].

While the research results on throughput capacity have greatly advanced our understanding of the maximum load accommodation of wireless networks, another equally important performance indicator, the packet delay, has received less attention. In the QoS-sensitive communications, the delay perceived by a packet is more QoS relevant than the total network throughput. The packet delay is the combinative result of various components that can be categorized into the bandwidth-incurred delay and the load-incurred delay. In this paper we are interested in determining the lower bound on the bandwidth-incurred delay, which is the transmission time spent by a packet in all the links along its transportation path. As a means to interpret the minimum packet transmission delay, we define the metric *Information Propagation Speed* and find out its upper bound.

In the pioneering paper [7], Zheng shows that there is a constant upper bound W on the information diffusion rate and a constant diffusion rate is achievable, regardless of the network population, in both the *extended* and the *dense* networks. Achieving W requires three conditions: i) every node uses an optimal transmission radius R, ii) the transportation distance of a packet is a multiple of R, and iii) the relay nodes are aligned with separation distance R. Lacking any of these conditions results in W unreachable.

However, a few interesting questions remain unanswered yet. First, if the packet transportation distance is known and not equal to a multiple of R, what is the best propagation strategy for the packet to achieve the fastest delivery? Since W is unreachable, is there a tighter speed upper bound? Second, when delivering a packet, we care about both *how fast* and *how well* the packet is delivered, that is, whether all the intended recipients can receive the packet successfully. When the network connectivity and the packet delivery satisfaction are considered, what is the speed upper bound under this constraint? Third, if the optimal transmission radius R is used but the relay nodes are not perfectly located, what is the gap between the actually achieved speed and the desired upper bound W? We attempt to provide the answers to these three questions in this paper.

As the first contribution, we show that there is another optimal transmission radius other than R if the packet transportation distance is not a multiple of R. We note that in broadcast communications, as the locations of packet recipients may not be known in advance, R is the best transmission strategy. In unicast communications, however, the location of the packet recipient may be known. If the known transportation distance is not a multiple of R, another transmission radius that optimally fits the specific distance should be used. Interestingly, we find that there is a unified optimal transmission radius and speed upper bound in large wireless networks.

As the second contribution, we determine the speed upper bound under the constraint of guaranteeing a given level of packet delivery satisfaction. We examine two different noise models. In the first model, the noise in the network is determined by the environmental noise. We show that there exists a threshold node density, above which there is a constant speed upper bound. In the second model, interference is the determinant source of noise. We show that there also exists a threshold node density, above which, however, the speed upper bound decreases to zero.

As the third contribution, we quantify the gap between the actually achieved speed and the desired upper bound in random networks. We prove that a packet propagates omnidirectionally in large networks and the speed gap reduces as node density

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increases. Furthermore, we show that in both noise models, there exists a threshold node density, below which the gap is bounded by constants and above which the gap converges to zero exponentially as node density increases to infinity.

The outline of this paper is as follows. We formulate the problem of information propagation speed in Section II. In Section III, we derive the unified speed upper bound for broadcast and unicast communications. The speed upper bound constrained by the packet delivery satisfaction is determined in Section IV, and the speed gap quantification is provided in Section V. Section VI concludes this paper.

II. Problem Formulation

Before we investigate the speed of information propagation, it is necessary to understand how information propagates in multihop wireless networks. First, we describe the network model used in this paper.

A. Network Model

We study a square-shaped network of n nodes in a large area $\mathcal{B} = [-\frac{l}{2}, \frac{l}{2}]^2$. The following assumptions are made in this paper regarding the node locations and communications.

- The nodes are static and randomly distributed obeying a Poisson point process with density λ.
- All the nodes share a B Hz available frequency band.
- Any two nodes can communicate over the direct link between them. The link is characterized by a path loss model with attenuation exponent $\alpha \geq 2$ [13] and its bandwidth is subject to the *Shannon Capacity*: $C = B\log_2(1+\text{SNR})$, where SNR is the signal-to-noise ratio. Advanced coding algorithms are used such that the link bandwidth approximates the Shannon capacity [14].
- Since higher transmission power gains larger link bandwidth but imposes stronger interference on other nodes, we assume that a uniform transmission power P is used by every node for fairness.
- The noise N is the sum of the ambient noise N_A and the interference noise N_I. For tractable modeling and analysis, we consider two cases in this paper: i) the ambience-dominant noise model ($N_A \gg N_I$, thus $N = N_A$), and ii) the interference-dominant noise model ($N_I \gg N_A$, thus $N = N_I$). In both models, N is assumed to be constant everywhere in the network.
- No directional antenna is used and no large signal-blocking obstacle exists in the network.
- The length of a packet is L bits.

For clarity, we make a few comments on these assumptions.

- All the distances in this paper are the Euclidean distance.
- Given two nodes v_i and v_j separated by a distance $d_{v_iv_j}$, the bandwidth of the direct link between them is $C_{v_iv_j} = B\log_2(1+\frac{P}{N}d_{v_iv_j}^{-\alpha})$.
- If v_i sends a packet to v_j at the full link capacity $C_{v_iv_j}$, a node v_k also receives the same packet if $d_{v_iv_k} \leq d_{v_iv_j}$, since $C_{v_iv_k} \geq C_{v_iv_j}$. On the other hand, if $d_{v_iv_k} > d_{v_iv_j}$, v_k does not receive the packet, as $C_{v_iv_k} < C_{v_iv_j}$. We define $r_{v_i} = d_{v_iv_j}$ as the *transmission radius* of v_i and $\mathcal{A}_{v_i} = \{s | d_{v_is} \leq r_{v_i}, s \in \mathcal{B}\}$ as the *coverage area* of v_i.

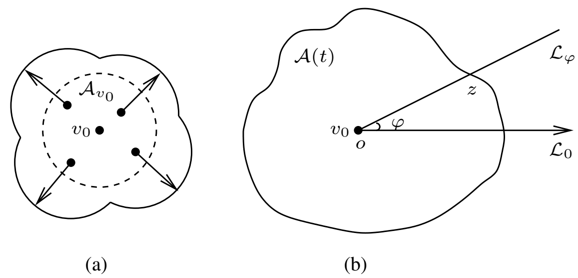


Fig. 1. Information propagation in multihop networks.

B. Information Propagation Speed

Information propagates in multihop wireless networks via rebroadcasting. An illustration is shown in Fig. 1, in which a packet is originated by node v_0. Node v_0 chooses a transmission radius r_{v_0} and broadcasts the packet. The packet is received and rebroadcast by all the neighbor nodes in $\mathcal{A}_{v_0}$. The rebroadcasting continues until the packet is received by its destination. Fig. 1(a) depicts the area reached by the packet after two hops.

Denote $\mathcal{V}(t)$ as the set of nodes that have received the packet by time t and $\widetilde{\mathcal{V}}(t) \subset \mathcal{V}(t)$ as the subset that have forwarded the packet by time t. The total area that the packet has reached by time t is expressed as $\mathcal{A}(t) = \cup_{v_i \in \widetilde{\mathcal{V}}(t)} \mathcal{A}_{v_i}$. In addition, denote $\mathcal{L}_\varphi$ as the line starting from v_0 in the direction $\varphi \in [0, 2\pi)$ and $\mathcal{L}_\varphi(t) = \mathcal{L}_\varphi \cap \mathcal{A}(t)$. In Fig. 1(b), $\mathcal{L}_\varphi(t)$ is the line segment $\overline{oz}$. The *Information Propagation Speed* in the direction φ is then defined to be

$$w_\varphi(t) = \frac{|\mathcal{L}_\varphi(t)|}{t} = \frac{\max_s\{d_{v_0s} | s \in \mathcal{L}_\varphi(t)\}}{t}. \quad (1)$$

Note that when node density is sufficiently large, $\mathcal{A}(t)$ is solid and $\mathcal{L}_\varphi(t)$ is continuous. When node density is small, holes may exist in $\mathcal{A}(t)$ and $\mathcal{L}_\varphi(t)$ may be fragmentary. In both cases, $|\mathcal{L}_\varphi(t)|$ by definition is the distance from v_0 to the farthest location reached by the packet in direction φ.

In the rest of this paper, we will formally derive the upper bound on $w_\varphi(t)$, examine the feasibility of the upper bound under the constraint of packet delivery satisfaction, and determine the gap between $w_\varphi(t)$ and its upper bound.

III. The Upper Bound on Speed $w_\varphi(t)$

As discussed earlier, the speed upper bounds for broadcast and unicast communications are different. Information may propagate slower in unicast communications with a tighter upper bound than in broadcast communications. In this section, we formally derive these two upper bounds. We also show that these two bounds converge in large networks.

A. Broadcast Communications

Suppose by time t a packet originated at v_0 has reached the location z in direction φ, as shown in Fig. 2. Denote

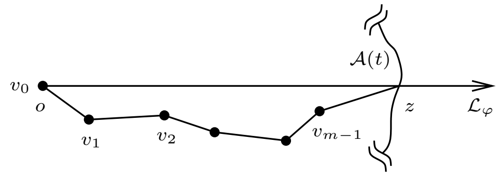


Fig. 2. The packet relay path in direction φ.

$\{v_0, v_1, \cdots, v_{m-1}\}$ as the relay path from o to z and $\tau_{v_i} = \frac{L}{B\log_2(1+\frac{P}{N}r_{v_i}^{-\alpha})}$ as the transmission duration of v_i. Then,

$$\begin{aligned} w_\varphi(t) &\le \frac{\sum_{i=0}^{m-2} d_{v_i v_{i+1}} + d_{v_{m-1}z}}{\sum_{i=0}^{m-1}\tau_{v_i}} \le \frac{\sum_{i=0}^{m-1} r_{v_i}}{\sum_{i=0}^{m-1}\tau_{v_i}} \\ &\le \max_i \frac{r_{v_i}}{\tau_{v_i}} \le \frac{B}{L}\max_r r\log_2(1+\frac{P}{N}r^{-\alpha}). \end{aligned}$$

The maximum of $r\log_2(1+\frac{P}{N}r^{-\alpha})$ occurs when $r = R_b = (\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$, where $y(\alpha)$ is the non-zero root of the equation

$$(1+y)\log_2(1+y) = \frac{\alpha}{\ln 2}y. \tag{2}$$

Thus,

$$w_\varphi(t) \le W_b = \frac{B}{L}R_b\log_2(1+\frac{P}{N}R_b^{-\alpha}). \tag{3}$$

If B, L, P, N and α are constant, R_b and W_b are constant. This is the same result as Zheng has obtained in [7]. The conditions for $w_\varphi(t) = W_b$, also discussed in [7], require: i) every relay node uses the optimal transmission radius R_b, ii) relay nodes are aligned and separated from each other by distance R_b, and iii) the distance from v_0 to the destination node v_d (or the farthest recipient node in broadcast communications) is a multiple of R_b. In broadcast communications, as the number of potential recipients may be very large and their locations may not be known in advance, using the transmission radius R_b is the best strategy for the fastest packet delivery.

B. Unicast Communications

In unicast communications, since there is only one destination, the source and the relay nodes may be aware of its location. If the distance between the source and the destination $d_{v_0v_d}$ is not a multiple of R_b, W_b is not achievable. In this case, we show that there is a tighter upper bound $W_u < W_b$ on $w_\varphi(t)$, as specified in the following theorem.

Theorem 1: There exists $d^* = (\frac{P}{N(2^\alpha-2)})^{\frac{1}{\alpha}}$ such that:

i) if $d_{v_0v_d} < d^*$, direct transmission from v_0 to v_d achieves the fastest speed;
ii) if $d_{v_0v_d} > d^*$, $w_\varphi(t) \le W_u = \frac{B}{L}R_u\log_2(1+\frac{P}{N}R_u^{-\alpha})$, where $R_u = d_{v_0v_d}/\mathrm{G}(d_{v_0v_d}/R_b)$ and function $\mathrm{G}(\cdot)$ rounds $d_{v_0v_d}/R_b$ to the nearest integer.

Before proving Theorem 1, we introduce a few notations and lemmas. Denote $t_{v_0v_d}$ as the transmission time from v_0 to v_d via any straightline relay path, $\mathcal{P}_{v_0v_d}^{(m)} = \{v_0, v_1, \cdots, v_{m-1}, v_d\}$ $(m \ge 1)$ as an m-hop straightline relay path from v_0 to v_d, and $t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m)})$ as the transmission time along $\mathcal{P}_{v_0v_d}^{(m)}$. We have the following lemmas.

Lemma 1: Consider $\mathcal{P}_{v_0v_d}^{(1)}$ and $\mathcal{P}_{v_0v_d}^{(2)}$. Define $\min t_{v_0v_d} = \min\{t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(1)}), t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(2)})\}$, then

$$\min t_{v_0v_d} = \begin{cases} \frac{L}{B\log_2(1+\frac{P}{N}d_{v_0v_d}^{-\alpha})} & \text{if } d_{v_0v_d} < d^*, \\ \frac{2L}{B\log_2(1+\frac{P}{N}(\frac{d_{v_0v_d}}{2})^{-\alpha})} & \text{if } d_{v_0v_d} > d^*. \end{cases}$$

Proof: Define $t(x) = \frac{L}{B\log_2(1+\frac{P}{N}x^{-\alpha})}$ and $t_{v_0v_d}(x) = t(x) + t(d_{v_0v_d} - x)$ $(0 \le x \le d_{v_0v_d})$. When $x = 0$ or $x = d_{v_0v_d}$, it is a 1-hop transmission. Otherwise, it is a 2-hop transmission. Function $t(x)$ has these properties:

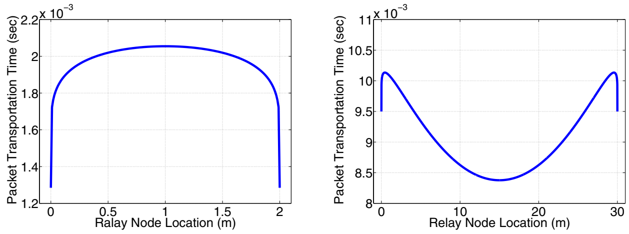
$$t'(x) = \frac{(\ln 2)\alpha LPx^{-\alpha-1}}{BN(1+\frac{P}{N}x^{-\alpha})\ln^2(1+\frac{P}{N}x^{-\alpha})} > 0,$$

$$\begin{aligned} t''(x) = &\frac{\alpha LPx^{-\alpha-2}}{(\ln 2)^2BN(1+\frac{P}{N}x^{-\alpha})^2\log_2^3(1+\frac{P}{N}x^{-\alpha})} \\ &\cdot[2\alpha\frac{P}{N}x^{-\alpha} - (\alpha+1+\frac{P}{N}x^{-\alpha})\ln(1+\frac{P}{N}x^{-\alpha})]. \end{aligned}$$

Define $y = \frac{P}{N}x^{-\alpha}$ and $f(y) = 2\alpha y - (\alpha+1+y)\ln(1+y)$. We have $f'(y) = 2\alpha - [\ln(1+y) + \frac{\alpha}{1+y} + 1]$. It is not difficult to find the following properties of $f'(y)$: i) $f'(0) = \alpha - 1 > 0$, ii) $f'(y)$ increases monotonically when $y \in [0, \alpha-1)$, iii) $f'(y)$ decreases monotonically when $y \in (\alpha-1, \infty)$, and iv) $\lim_{y\to\infty} f'(y) = -\infty$. These properties indicate the existence of $y_1 > 0$ such that $f'(y) > 0$ when $y \in [0, y_1)$ and $f'(y) < 0$ when $y \in (y_1, \infty)$. Since $f(0) = 0$, there must exist $y_2 > y_1$ such that $f(y) > 0$ when $y \in (0, y_2)$ and $f(y) < 0$ when $y \in (y_2, \infty)$. Define $x_2 = (\frac{P}{Ny_2})^{\frac{1}{\alpha}}$ and $f(x) = 2\alpha\frac{P}{N}x^{-\alpha} - (\alpha + 1 + \frac{P}{N}x^{-\alpha})\ln(1+\frac{P}{N}x^{-\alpha})$. Then $f(x) < 0$ when $x \in (0, x_2)$ and $f(x) > 0$ when $x \in (x_2, \infty)$. Besides, since $\forall x > 0$, $\frac{\alpha LPx^{-\alpha-2}}{(\ln 2)^2BN(1+\frac{P}{N}x^{-\alpha})^2\log_2^3(1+\frac{P}{N}x^{-\alpha})} > 0$, we have $t''(x) < 0$ (i.e. $t(x)$ is strictly concave) when $x \in (0, x_2)$ and $t''(x) > 0$ (i.e. $t(x)$ is strictly convex) when $x \in (x_2, \infty)$.

Note that $t_{v_0v_d}(x)$ is symmetric with respect to $x = \frac{d_{v_0v_d}}{2}$. Besides, $t'_{v_0v_d}(\frac{d_{v_0v_d}}{2}) = 0$ and $t''_{v_0v_d}(0) = t''_{v_0v_d}(d_{v_0v_d}) < 0$ (because $t''(0) = -\infty$, $t''(d_{v_0v_d}) < \infty$, $t''_{v_0v_d}(0) = t''_{v_0v_d}(d_{v_0v_d}) = t''(0) + t''(d_{v_0v_d}) = -\infty$). Next, we discuss the minimum of $t_{v_0v_d}(x)$ in three cases.

i) $d_{v_0v_d} \in (0, x_2]$. $t(x)$ and $t(d_{v_0v_d} - x)$ are concave on $[0, d_{v_0v_d}]$, so $t_{v_0v_d}(x)$ is concave with no local minimum.
ii) $d_{v_0v_d} \in (x_2, 2x_2]$. $t(x)$ is concave on $[0, x_2]$ and convex on $[x_2, d_{v_0v_d}]$. $t(d_{v_0v_d} - x)$ is convex on $[0, d_{v_0v_d} - x_2]$ and concave on $[d_{v_0v_d} - x_2, d_{v_0v_d}]$. So, $t_{v_0v_d}(x)$ must be concave on $[d_{v_0v_d} - x_2, x_2]$, while either concave or convex on $[0, d_{v_0v_d} - x_2] \cup [x_2, d_{v_0v_d}]$. However, $t''_{v_0v_d}(0) = t''_{v_0v_d}(d_{v_0v_d}) < 0$ indicates concavity. Thus, $t_{v_0v_d}(x)$ is concave on $[0, d_{v_0v_d}]$ with no local minimum.
iii) $d_{v_0v_d} \in (2x_2, \infty)$. Similar to the discussion in case ii), $t_{v_0v_d}(x)$ must be convex on $[x_2, d_{v_0v_d} - x_2]$, while either concave or convex on $[0, x_2] \cup [d_{v_0v_d} - x_2, d_{v_0v_d}]$. Again, $t''_{v_0v_d}(0) = t''_{v_0v_d}(d_{v_0v_d}) < 0$ indicates concavity. Hence, $t_{v_0v_d}(x)$ has one local minimum at $x = \frac{d_{v_0v_d}}{2}$ and two local maxima in $[0, x_2]$ and $[d_{v_0v_d} - x_2, d_{v_0v_d}]$.



(a) $d_{v_0v_d} = 2$m (b) $d_{v_0v_d} = 30$m

Fig. 3. Two examples of the function $t_{v_0v_d}(x)$, $B = 100$ KHz, $L = 1024$ bits, $\frac{P}{N} = 10^3$, $\alpha = 2$.

Fig. 3 visualizes the function $t_{v_0v_d}(x)$. In Fig. 3(a), $d_{v_0v_d} \in (0, 2x_2]$, $t_{v_0v_d}(x)$ is concave. It does not have local minimum. In Fig. 3(b), $d_{v_0v_d} \in (2x_2, \infty)$, $t_{v_0v_d}(x)$ is convex in the middle while concave near the two ends. It has one local minimum. Summarizing all the three cases, $t_{v_0v_d}(x)$ has at most one local minimum which occurs at $x = \frac{d_{v_0v_d}}{2}$. Hence,

$$\begin{aligned}\min t_{v_0v_d} &= \min\{t_{v_0v_d}(0), t_{v_0v_d}(\frac{d_{v_0v_d}}{2}), t_{v_0v_d}(d_{v_0v_d})\} \\ &= \begin{cases} \frac{L}{B\log_2(1+\frac{P}{N}d_{v_0v_d}^{-\alpha})} & \text{if } d_{v_0v_d} < d^*, \\ \frac{2L}{B\log_2(1+\frac{P}{N}(\frac{d_{v_0v_d}}{2})^{-\alpha})} & \text{if } d_{v_0v_d} > d^*. \end{cases}\end{aligned}$$

■

Lemma 1 states the fact that: i) if $d_{v_0v_d} < d^*$, 1-hop direct transmission is faster than any 2-hop relay transmission; ii) if $d_{v_0v_d} > d^*$, choosing a relay node at the exact middle location between the source and the destination results in the fastest transmission among all the 2-hop relay paths, and it is also faster than the 1-hop direct transmission; iii) if $d_{v_0v_d} = d^*$, 1-hop direct transmission is as fast as the 2-hop relay transmission with the relay node at the exact middle location, and both are faster than any other 2-hop transmissions.

Lemma 2: Consider all the m-hop relay paths $\mathcal{P}_{v_0v_d}^{(m)}$ ($m \geq 1$). Define $\min t_{v_0v_d} = \min_m\{t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m)})\}$. If $d_{v_0v_d} < d^*$, $\min t_{v_0v_d} = t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(1)})$. That is, 1-hop direct transmission is faster than any multihop transmissions.

Proof: By Lemma 1, if $d_{v_0v_d} < d^*$, $t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(1)}) < t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(2)})$. $\forall\ m \geq 2$, as $d_{v_{m-1}v_d} < d_{v_{m-2}v_d} < \cdots < d_{v_1v_d} < d_{v_0v_d} < d^*$, apply the result of Lemma 1 recursively,

$$\begin{aligned}t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(1)}) &< \sum_{i=0}^{0} t_{v_iv_{i+1}}(\mathcal{P}_{v_iv_{i+1}}^{(1)}) + t_{v_1v_d}(\mathcal{P}_{v_1v_d}^{(1)}) \\ &\vdots \\ &< \sum_{i=0}^{m-2} t_{v_iv_{i+1}}(\mathcal{P}_{v_iv_{i+1}}^{(1)}) + t_{v_{m-1}v_d}(\mathcal{P}_{v_{m-1}v_d}^{(1)}) \\ &= t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m)}).\end{aligned}$$

■

Lemma 3: Consider all the m-hop relay path $\mathcal{P}_{v_0v_d}^{(m)}$ ($m \geq 1$). Define $\min t_{v_0v_d} = \min_m\{t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m)})\}$. If $d_{v_0v_d} > d^*$, $\min t_{v_0v_d} = \min_m\{t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m),e})\}$, in which $\mathcal{P}_{v_0v_d}^{(m),e} = \{v_0, v_1, \cdots, v_{m-1}, v_d\}$ and $d_{v_0v_1} = \cdots = d_{v_{m-1}v_d}$. That is, the fastest transmission must be achieved along a relay path in which the relay nodes are separated equally.

Proof: It is equivalent to show that $\forall\ \mathcal{P}_{v_0v_d}^{(m)}$ ($m \geq 1$), $\exists\ \mathcal{P}_{v_0v_d}^{(m'),e} = \{v_0, v_1, \cdots, v_{m'-1}, v_d\}$, where $d_{v_0v_1} = d_{v_1v_2} = \cdots = d_{v_{m'-1}v_d}$, such that $t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m'),e}) \leq t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m)})$.

To prove the existence of $\mathcal{P}_{v_0v_d}^{(m'),e}$, we consider the following node removal and relocation process on any $\mathcal{P}_{v_0v_d}^{(m)}$. For each node $v_i \in \mathcal{P}_{v_0v_d}^{(m)}$, we make the two changes below in sequence:

1) Node removal. Find the set of nodes $\{v_j \mid d_{v_iv_j} \leq d^*, j = i+1, \cdots, i+k\}$. If $k > 1$, remove the nodes $\{v_j, j = i+1, \cdots, i+k-1\}$ from $\mathcal{P}_{v_0v_d}^{(m)}$.
2) Node relocation. If v_i is the last relay node or $v_i = v_d$, skip this step. Otherwise, if $k = 0$ (k is the number of nodes found in step 1), relocate v_{i+1} to the exact middle point between v_i and v_{i+2}; if $k > 0$, relocate v_{i+k} to the exact middle point between v_i and v_{i+k+1}.

This process initiates at v_0, proceeds node by node toward v_d, and iterates after v_d until there is no more node removal and no more node relocation in the resulting relay path $\mathcal{P}_{v_0v_d}^{(m'),e}$.

First, we show that the resulting path $\mathcal{P}_{v_0v_d}^{(m'),e}$ has a shorter transmission time than the original path $\mathcal{P}_{v_0v_d}^{(m)}$. In the node removal step, since $d_{v_iv_{i+k}} \leq d^*$, by Lemma 2, 1-hop direct transmission from v_i to v_{i+k} is faster than the k-hop transmission via v_{i+1}, $\cdots$, v_{i+k-1}. Therefore, removing $\{v_j, j = i+1, \cdots, i+k-1\}$ results in faster transmission. In the node relocation step, because $d_{v_iv_{i+2}} > d^*$ (if $k = 0$) or $d_{v_iv_{i+k+1}} > d^*$ (if $k > 0$), by Lemma 1, relocation of v_{i+1} (if $k = 0$) or v_{i+k} (if $k > 0$) results in faster transmission. Therefore, $\mathcal{P}_{v_0v_d}^{(m'),e}$ has shorter transmission time than $\mathcal{P}_{v_0v_d}^{(m)}$.

Second, we prove that in the resulting path $\mathcal{P}_{v_0v_d}^{(m'),e}$, $d_{v_0v_1} = \cdots = d_{v_{m'-1}v_d}$. Because the node removal step takes relay nodes away and the number of remaining relay nodes must be non-negative, it is obvious that the number of relay nodes converges to a value m' ($0 \leq m' \leq m$). After that, there are no more removals, but relocations may continue. As the transmission time from v_0 to v_d decreases during the relocations (proven above) and it is non-negative, it must converge to some value, after which there are no more relocations. If $d_{v_0v_1}$, $\cdots$, $d_{v_{m'-1}v_d}$ are not all equal, relocation will continue. Thus, they must be all equal by the end of the relocation process.

Finally, it is obvious that m' can be replaced by m safely, so $\min t_{v_0v_d} = \min_m\{t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m),e})\}$. ■

We are now ready to prove Theorem 1 as follows.

Proof: The statement i) is already proven in Lemma 2. The statement ii) considers $d_{v_0v_d} > d^*$. In this case, Lemma 3 shows $\min t_{v_0v_d} = \min_m\{t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m),e})\}$. Expressing $t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m),e}) = mL/B\log_2(1+\frac{P}{N}(\frac{d_{v_0v_d}}{m})^{-\alpha})$ and solving $\mathrm{d}t_{v_0v_d}(\mathcal{P}_{v_0v_d}^{(m),e})/\mathrm{d}m = 0$, we obtain the optimal number of relay hops $m = \mathrm{G}(d_{v_0v_d}/(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}) = \mathrm{G}(d_{v_0v_d}/R_b)$, the optimal transmission radius $R_u = d_{v_0v_d}/\mathrm{G}(d_{v_0v_d}/R_b)$, and the upper bound on the information propagation speed

$$w_\varphi(t) \le W_u = \frac{B}{L} R_u \log_2(1 + \frac{P}{N} R_u^{-\alpha}). \quad (4)$$

Because R_b is the unique maximizer for $r \log_2(1 + \frac{P}{N} r^{-\alpha})$, $W_u < W_b$ if $R_u \neq R_b$. ■

Unlike R_b and W_b, R_u and W_u are determined not only by B, L, P, N and α, but also by $d_{v_0 v_d}$. The conditions for $w_\varphi(t) = W_u$ are: i) every relay node uses the optimal transmission radius R_u, and ii) the relay nodes are aligned and separated by distance R_u. Note that $\lim_{d_{v_0 v_d} \to \infty} R_u = R_b$, indicating that the two bounds converge in large networks.

IV. The Feasibility-Constrained Upper Bound

We have shown that there is a unified upper bound on $w_\varphi(t)$ for broadcast and unicast communications in large networks: $W_b = W_u = W = \frac{B}{L} R \log_2(1 + \frac{P}{N} R^{-\alpha})$, where $R = (\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$. In this section, we study the *feasibility* of this upper bound. As the nodes are randomly located, it is possible that the network is not connected by using the transmission radius R. If the destination node cannot be reached by using R, the maximum propagation speed W becomes infeasible because the packet is undeliverable. Therefore, we need to understand the maximum information propagation speed under the constraint of packet delivery satisfaction.

We define the term *γ-feasible delivery* to provide a measurement on the degree of packet delivery satisfaction. The delivery of a packet is γ-feasible if the packet can reach all the intended recipients with a probability no less than γ ($0 \le \gamma \le 1$). Subsequently, we define a transmission radius r to be γ-feasible if this r provides γ-feasible delivery and a speed upper bound W (denoted as W_γ) to be γ-feasible if it is the maximum speed that guarantees γ-feasible delivery. Obviously, given a γ-feasible r, any transmission radius larger than r is also γ-feasible. However, since $r \log_2(1 + \frac{P}{N} r^{-\alpha})$ is a decreasing function on $r \in [R, \infty)$, W_γ is achieved at the γ-feasible r (denoted as R_γ) that is the closest to R.

In the next, we study W_γ in variable node densities. Before the investigation, we first cite a relevant result that will be referenced later. Penrose shows in [15] that the longest edge M_n in the minimal spanning tree over n Poisson distributed random nodes in a unit square satisfies

$$\lim_{n\to\infty} \Pr[n\pi M_n^2 - \log(n) \le \beta] = \exp(-e^{-\beta}).$$

Zheng [7] further proves that in an extended network with unit node density, if $\lim_{n\to\infty} c(n) = \infty$,

$$\lim_{n\to\infty} \Pr[-c(n) \le \pi M_n^2 - \log(n) \le c(n)] = 1.$$

Scaling the extended network to the dense network, we have

$$\lim_{n\to\infty} \Pr[-c(n) \le n\pi M_n^2 - \log(n) \le c(n)] = 1.$$

Choosing $c(n) = \epsilon \log(n)$ ($\epsilon > 0$) and replacing n with λ ($n = \lambda$ in a unit square), we have

$$\lim_{\lambda\to\infty} \Pr\Big[\sqrt{\frac{(1-\epsilon)\log(\lambda)}{\lambda\pi}} \le M_\lambda \le \sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}}\Big] = 1. \quad (5)$$

Our network may be viewed as the tiles of unit squares with node density λ over an area $\mathcal{B}$. Thus Equation (5) applies to our network model too. Next, we discuss W_γ with variable node densities in two different noise models.

A. The Ambience-Dominant Noise Model

In the ambience-dominant noise model, the noise N is determined by the ambient noise N_A, which is irrelevant to the node density λ. The γ-feasible speed upper bound W_γ is given by the following theorem.

Theorem 2: In the ambience-dominant noise model ($N = N_A$), given the feasibility parameter γ, there exists a threshold node density λ_A such that: i) if $\lambda < \lambda_A$, $R_\gamma(\lambda) = R_A\sqrt{\frac{\lambda_A}{\lambda}}$ and $W_\gamma(\lambda) = \frac{B}{L}\big(R_A\sqrt{\frac{\lambda_A}{\lambda}}\big)\log_2\big(1 + \frac{P}{N}\big(R_A\sqrt{\frac{\lambda_A}{\lambda}}\big)^{-\alpha}\big)$, where $R_A = (\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$, and ii) if $\lambda > \lambda_A$, $R_\gamma(\lambda) = \big(\frac{P}{Ny(\alpha)}\big)^{\frac{1}{\alpha}}$ and $W_\gamma(\lambda) = \frac{B}{L}\big(\frac{P}{Ny(\alpha)}\big)^{\frac{1}{\alpha}}\log_2(1 + y(\alpha))$.

Proof: By Equation (5), $\exists\ \lambda_A^{(1)}$ s.t. $\forall\ \lambda \ge \lambda_A^{(1)}$,

$$\Pr\Big[M_\lambda \le \sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}}\Big] \ge \gamma.$$

Let $\lambda_A^{(2)}$ denote the biggest root of the equation

$$\sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}} = \Big(\frac{P}{Ny(\alpha)}\Big)^{\frac{1}{\alpha}}.$$

If this equation has no real root, define $\lambda_A^{(2)} = 0$. We see that, $\forall\ \lambda \ge \lambda_A^{(2)}$, $\sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}} \le (\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$, since $\lim_{\lambda\to\infty}\sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}} = 0$. Denoting $\lambda_A = \max\{\lambda_A^{(1)}, \lambda_A^{(2)}\}$, we have $\forall\ \lambda > \lambda_A$, $\Pr[M_\lambda \le (\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}] \ge \gamma$. This is to say that when $\lambda > \lambda_A$ a packet can be delivered to every node in the network with probability no less than γ by using the optimal transmission radius $(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$ and thus $(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$ is γ-feasible. In this case, $W_\gamma(\lambda) = \frac{B}{L}(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}\log_2(1 + \frac{P}{N}(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}(-\alpha)}) = \frac{B}{L}\big(\frac{P}{Ny(\alpha)}\big)^{\frac{1}{\alpha}}\log_2(1 + y(\alpha))$. This proves the statement ii). When $\lambda < \lambda_A$, $(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$ is not γ-feasible. Since $(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$ is the smallest γ-feasible transmission radius in the network with node density λ_A and scaling the area of this network by a factor $\frac{\lambda_A}{\lambda}$ results in another network with node density λ, $(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}\sqrt{\frac{\lambda_A}{\lambda}}$ must be the smallest (also the closest to the optimal radius $(\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$) γ-feasible transmission radius in the network with node density λ. As such, when $\lambda < \lambda_A$, $W_\gamma(\lambda) = \frac{B}{L}\big(R_A\sqrt{\frac{\lambda_A}{\lambda}}\big)\log_2\big(1 + \frac{P}{N}\big(R_A\sqrt{\frac{\lambda_A}{\lambda}}\big)^{-\alpha}\big)$, where $R_A = (\frac{P}{Ny(\alpha)})^{\frac{1}{\alpha}}$. This proves the statement i). ■

B. The Interference-Dominant Noise Model

In the interference-dominant noise model, the noise N is determined by the interference N_I, which is a linear function of the node density λ, as proven in the following lemma.

Lemma 4: In a network where the nodes are randomly distributed in a Poisson point process with density λ and a uniform transmission power P is used by every node, the interference at any location is $N_I(\lambda) = \lambda N_I(1)$, where $N_I(1)$ is the interfence at this location when $\lambda = 1$.

Proof: Choosing two arbitrary locations $z_1, z_2 \in \mathcal{B}$ and defining N_{I,z_1,z_2} as the interference at z_1 caused by the transmissions at z_2, we have

$$\begin{aligned} N_{I,z_1,z_2}(\lambda) &= \lim_{\delta\to 0} \frac{N_{I,z_1,z_2(\delta)}(\lambda)}{\delta} \\ &= \lim_{\delta\to 0} \frac{\sum_{k=0}^{\infty} e^{-\lambda\delta}\frac{(\lambda\delta)^k}{k!}\cdot k\cdot P d_{z_1 z_2}^{-\alpha}}{\delta} \\ &= \lambda\cdot P d_{z_1 z_2}^{-\alpha} \\ &= \lambda\cdot N_{I,z_1,z_2}(1), \end{aligned}$$

where δ is a small area around z_2 and $N_{I,z_1,z_2(\delta)}$ is the interference from δ. The total interference at z_1 is

$$N_{I,z_1}(\lambda) = \int_{z_2\in\mathcal{B}} N_{I,z_1,z_2}(\lambda)\,\mathrm{d}z_2 = \lambda N_{I,z_1}(1).$$

As z_1 is arbitrary, $N_I(\lambda) = \lambda N_I(1)$. ■

Since $N = N_I$, we write $N(\lambda) = \lambda N(1)$. In this noise model, the optimal transmission radius has the form $(\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}}$. The γ-feasible speed upper bound W_γ is given by the theorem below.

Theorem 3: In the interference-dominant noise model ($N = N_I$), given the feasibility parameter γ, there exists a threshold node density λ_I such that: i) if $\lambda < \lambda_I$, $R_\gamma(\lambda) = R_I\sqrt{\frac{\lambda_I}{\lambda}}$ and $W_\gamma(\lambda) = \frac{B}{L}\big(R_I\sqrt{\frac{\lambda_I}{\lambda}}\big)\log_2\big(1 + \frac{P}{\lambda N(1)}\big(R_I\sqrt{\frac{\lambda_I}{\lambda}}\big)^{-\alpha}\big)$, where $R_I = (\frac{P}{\lambda_I N(1) y(\alpha)})^{\frac{1}{\alpha}}$, and ii) if $\lambda > \lambda_I$, $R_\gamma(\lambda) = \big(\frac{P}{\lambda N(1) y(\alpha)}\big)^{\frac{1}{\alpha}}$ and $W_\gamma(\lambda) = \frac{B}{L}\big(\frac{P}{\lambda N(1) y(\alpha)}\big)^{\frac{1}{\alpha}}\log_2(1 + y(\alpha))$.

Proof: Similar to Theorem 2, $\exists\ \lambda_I^{(1)}$, s.t. $\forall\ \lambda \geq \lambda_I^{(1)}$,

$$\Pr\Big[M_\lambda \leq \sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}}\Big] \geq \gamma.$$

Let $\lambda_I^{(2)}$ denote the biggest root of the equation

$$\sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}} = \Big(\frac{P}{\lambda N(1) y(\alpha)}\Big)^{\frac{1}{\alpha}}.$$

If this equation has no real root, define $\lambda_I^{(2)} = 0$. We see that $\forall\ \lambda \geq \lambda_I^{(2)}$, $\sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}} \leq (\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}}$, since $\lim_{\lambda\to\infty}\sqrt{\frac{(1+\epsilon)\log(\lambda)}{\lambda\pi}}\Big/(\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}} = 0$. By denoting $\lambda_I = \max\{\lambda_I^{(1)}, \lambda_I^{(2)}\}$, we then have $\forall\ \lambda > \lambda_I$, $\Pr[M_\lambda \leq (\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}}] \geq \gamma$. This shows that when $\lambda > \lambda_I$ the optimal transmission radius $(\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}}$ is γ-feasible. So, when $\lambda > \lambda_I$, $W_\gamma(\lambda) = \frac{B}{L}(\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}}\log_2(1 + \frac{P}{\lambda N(1)}(\frac{P}{\lambda N(1) y(\alpha)})^{\frac{1}{\alpha}(-\alpha)}) = \frac{B}{L}\big(\frac{P}{\lambda N(1) y(\alpha)}\big)^{\frac{1}{\alpha}}\log_2(1 + y(\alpha))$. This proves the statement ii). When $\lambda < \lambda_I$, similar to our discussion in Theorem 2, because $(\frac{P}{\lambda_I N(1) y(\alpha)})^{\frac{1}{\alpha}}$ is the smallest γ-feasible transmission radius for node density λ_I, $(\frac{P}{\lambda_I N(1) y(\alpha)})^{\frac{1}{\alpha}}\sqrt{\frac{\lambda_I}{\lambda}}$ is the smallest (thus the closest to the optimal radius $(\frac{P}{\lambda_I N(1) y(\alpha)})^{\frac{1}{\alpha}}$) γ-feasible transmission radius for node density λ. Therefore, when $\lambda < \lambda_I$, $W_\gamma(\lambda) = \frac{B}{L}\big(R_I\sqrt{\frac{\lambda_I}{\lambda}}\big)\log_2\big(1 + \frac{P}{\lambda N(1)}\big(R_I\sqrt{\frac{\lambda_I}{\lambda}}\big)^{-\alpha}\big)$, where $R_I = (\frac{P}{\lambda_I N(1) y(\alpha)})^{\frac{1}{\alpha}}$. This proves the statement i). ■

C. Comparison of the Two Noise Models

Interestingly, we find that the γ-feasible speed upper bound $W_\gamma(\lambda)$ behaves quite differently in these two noise models.

In the ambience-dominant noise model, when $\lambda < \lambda_A$, $W_\gamma(\lambda) = \frac{B}{L}\big(R_A\sqrt{\frac{\lambda_A}{\lambda}}\big)\log_2\big(1 + \frac{P}{N}\big(R_A\sqrt{\frac{\lambda_A}{\lambda}}\big)^{-\alpha}\big)$. Because $W_\gamma(\lambda)$ achieves its maximum at $\lambda = \lambda_A$, $W_\gamma(\lambda)$ is an increasing function of λ when $0 < \lambda < \lambda_A$. When $\lambda > \lambda_A$, $W_\gamma(\lambda) = \frac{B}{L}\big(\frac{P}{N y(\alpha)}\big)^{\frac{1}{\alpha}}\log_2(1 + y(\alpha))$, which is a constant. This is to say, given sufficiently large node density, there is a constant upper bound on the information propagation speed.

In the interference-dominant noise model, when $\lambda < \lambda_I$, $W_\gamma(\lambda) = \frac{B}{L}\big(R_I\sqrt{\frac{\lambda_I}{\lambda}}\big)\log_2\big(1 + \frac{P}{\lambda N(1)}\big(R_I\sqrt{\frac{\lambda_I}{\lambda}}\big)^{-\alpha}\big)$. $W_\gamma(\lambda)$ reaches its maximum at $\lambda = \lambda_I\big(\frac{y(\alpha-2)}{y(\alpha)}\big)^{\frac{1}{\frac{\alpha}{2}-1}}$, where $y(\alpha-2)$ is the non-zero root of the equation

$$(1+y)\log_2(1+y) = \frac{\alpha-2}{\ln 2}y. \tag{6}$$

If $2 \leq \alpha \leq 3$, $y(\alpha-2)$ does not exist. In this case $W_\gamma(\lambda)$ is a decreasing function of λ when $0 < \lambda < \lambda_I$. If $\alpha > 3$, $y(\alpha-2)$ exists and $y(\alpha-2) < y(\alpha)$, indicating $\lambda_I\big(\frac{y(\alpha-2)}{y(\alpha)}\big)^{\frac{1}{\frac{\alpha}{2}-1}} < \lambda_I$. Thus, $W_\gamma(\lambda)$ increases when $0 < \lambda < \lambda_I\big(\frac{y(\alpha-2)}{y(\alpha)}\big)^{\frac{1}{\frac{\alpha}{2}-1}}$ and decreases when $\lambda_I\big(\frac{y(\alpha-2)}{y(\alpha)}\big)^{\frac{1}{\frac{\alpha}{2}-1}} < \lambda < \lambda_I$. When $\lambda > \lambda_I$, $W_\gamma(\lambda) = \frac{B}{L}\big(\frac{P}{\lambda N(1) y(\alpha)}\big)^{\frac{1}{\alpha}}\log_2(1 + y(\alpha))$, which decreases to zero as λ approaches infinity. Therefore, in the interference-dominant noise model, information propagation becomes impossible when the node density is extremely large. The strong interference prevents the transmission of any packet.

V. The Gap between $w_\varphi(t)$ and $W_\gamma(\lambda)$

We have shown that, given the parameter γ, there exists an optimal transmission radius $R_\gamma(\lambda)$ that may achieve the maximum information propagation speed $W_\gamma(\lambda)$ in a network with node density λ. However, as we have discussed earlier, actually achieving this maximum speed requires an additional condition that all the relay nodes are aligned and separated from each other by the distance $R_\gamma(\lambda)$. Since the nodes are randomly distributed, it is impossible to find these perfectly located relay nodes when $\lambda < \infty$. There is always a gap between the actually achievable speed $w_\varphi(t)$ and the bound $W_\gamma(\lambda)$. We quantify this gap in this section.

By definition, the actual information propagation speed is measured by $w_\varphi(t) = \frac{|\mathcal{L}_\varphi(t)|}{t}$. Due to the randomness of node locations, this speed may be faster or slower when the packet travels through different subareas in the network. To evaluate $w_\varphi(t)$ without introducing the subarea bias, we define the *long-term speed* in the direction φ to be

$$w_\varphi = \lim_{t\to\infty} w_\varphi(t) = \lim_{t\to\infty}\frac{|\mathcal{L}_\varphi(t)|}{t}. \tag{7}$$

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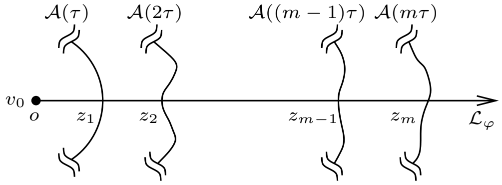


Fig. 4. Information propagation in multihops in direction φ.

Since every node uses the same optimal transmission radius $R_\gamma(\lambda)$, the 1-hop transmission time $\tau = \frac{L}{B\log_2(1+\frac{P}{N}R_\gamma^{-\alpha}(\lambda))}$ is the same for every node. Thus, Equation (7) is rewritten as

$$w_\varphi = \lim_{m\to\infty}\frac{Z_m}{m\tau} = \lim_{m\to\infty}\frac{\sum_{i=1}^m \rho_i}{m\tau} = \frac{\overline{\rho}}{\tau}, \tag{8}$$

where $Z_i = d_{oz_i}$, $\rho_i = Z_i - Z_{i-1}$ and $\overline{\rho} = \lim_{m\to\infty}\frac{\sum_{i=1}^m \rho_i}{m} = E[\rho_i]$, as shown in Fig. 4.

First, we show that the actual information propagation speed is omnidirectional in large networks. In the long term, a packet is disseminated to the same distance away in any direction and the frontier of propagation is in a circular shape centered at the source node, as specified in the following theorem.

Theorem 4: In a network with homogeneous node distributions, $\forall\ \varphi_1, \varphi_2 \in [0, 2\pi)$, $w_{\varphi_1} = w_{\varphi_2} = w$.

Proof: By definition, $w_\varphi = \frac{\overline{\rho}}{\tau}$. All we need to show is $\overline{\rho}_{\varphi_1} = \overline{\rho}_{\varphi_2}$. As the nodes are distributed homogeneously, the propagation distances in φ_1 and φ_2 after i hops, Z_{i,φ_1} and Z_{i,φ_2}, are two random variables with the same probability distribution. For the same reason Z_{i-1,φ_1} and Z_{i-1,φ_2} also have the same probability distribution. Since $\rho_{i,\varphi_1} = Z_{i,\varphi_1} - Z_{i-1,\varphi_1}$ and $\rho_{i,\varphi_2} = Z_{i,\varphi_2} - Z_{i-1,\varphi_2}$, ρ_{i,φ_1} and ρ_{i,φ_2} must have the same probability distribution. Therefore, $\overline{\rho}_{\varphi_1} = E[\rho_{i,\varphi_1}] = E[\rho_{i,\varphi_2}] = \overline{\rho}_{\varphi_2}$. ■

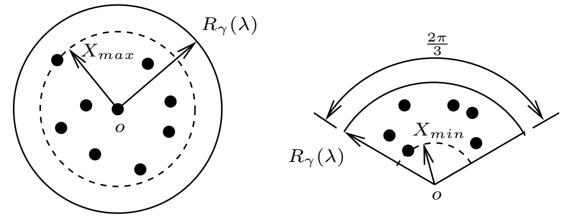
As we will show next that w is determined by the node density λ, we write $w = w(\lambda) = \frac{\overline{\rho}(\lambda)}{\tau}$. We define the gap between the actual speed $w(\lambda)$ and its upper bound $W_\gamma(\lambda)$ as

$$\varepsilon(\lambda) = \frac{W_\gamma(\lambda) - w(\lambda)}{W_\gamma(\lambda)} = \frac{R_\gamma(\lambda) - \overline{\rho}(\lambda)}{R_\gamma(\lambda)}. \tag{9}$$

Theorem 5: In a network where the nodes are randomly distributed in a Poisson point process with density λ, $\forall\ \lambda_1 < \lambda_2$, $\varepsilon(\lambda_1) > \varepsilon(\lambda_2)$. That is, $\varepsilon(\lambda)$ is a strictly decreasing function of λ.

Proof: By definition, $\varepsilon(\lambda) = 1 - \frac{\overline{\rho}(\lambda)}{R_\gamma(\lambda)}$. To prove $\forall\ \lambda_1 < \lambda_2$, $\varepsilon(\lambda_1) > \varepsilon(\lambda_2)$, it is equivalent to show $\frac{\overline{\rho}(\lambda_1)}{R_\gamma(\lambda_1)} < \frac{\overline{\rho}(\lambda_2)}{R_\gamma(\lambda_2)}$.

First, we show $\overline{\rho}(\lambda_1) < \overline{\rho}(\lambda_2)$. We start with a network of node density λ_2. Suppose a packet originated by node v_0 has propagated over a distance of $Z_m(\lambda_2)$ to reach location $z_m(\lambda_2)$ in an arbitrary direction φ after m hops and denote $\mathcal{P} = \{v_0, v_1, \cdots, v_{m-1}\}$ as the m-hop relay path travelled through by the packet to reach $z_m(\lambda_2)$. Now reduce the node density to λ_1 by randomly removing each node (except v_0) from the network with probability $\frac{\lambda_2-\lambda_1}{\lambda_2}$. From the properties of Poisson process, we know that the nodes in the resulting network are Poisson distributed with density λ_1. Since removing any $v_i \in \{v_1, v_2, \cdots, v_{m-1}\}$ disrupts $\mathcal{P}$, the survival probability of $\mathcal{P}$ is



(a) (b)

Fig. 5. Definitions of X_{max} and X_{min}.

$$\Pr[\mathcal{P}\text{ survives}] = \left(\frac{\lambda_1}{\lambda_2}\right)^{m-1}.$$

When $m \to \infty$, $\Pr[\mathcal{P}\text{ survives}] \to 0$, implying $z_m(\lambda_2)$ is unreachable in the resulting network. Denoting $Z_m(\lambda_1)$ as the propagation distance of the packet in direction φ after m hops in the resulting network, we have $Z_m(\lambda_1) < Z_m(\lambda_2)$ as $m \to \infty$, which gives

$$\overline{\rho}(\lambda_1) = \lim_{m\to\infty}\frac{Z_m(\lambda_1)}{m} < \lim_{m\to\infty}\frac{Z_m(\lambda_2)}{m} = \overline{\rho}(\lambda_2).$$

Next, we show $R_\gamma(\lambda_1) \geq R_\gamma(\lambda_2)$. In the ambience-dominant noise model, $R_\gamma(\lambda) = R_A\sqrt{\frac{\lambda_A}{\lambda}}$ when $\lambda < \lambda_A$, and $R_\gamma(\lambda) = \left(\frac{P}{Ny(\alpha)}\right)^{\frac{1}{\alpha}}$ when $\lambda > \lambda_A$. For all λ, $R_\gamma(\lambda)$ is a decreasing function (not strictly). In the interference-dominant noise model, $R_\gamma(\lambda) = R_I\sqrt{\frac{\lambda_I}{\lambda}}$ when $\lambda < \lambda_I$, and $R_\gamma(\lambda) = \left(\frac{P}{\lambda N(1)y(\alpha)}\right)^{\frac{1}{\alpha}}$ when $\lambda > \lambda_I$. For all λ, $R_\gamma(\lambda)$ is a strictly decreasing function. So, $R_\gamma(\lambda_1) \geq R_\gamma(\lambda_2)$ in both models.

Combining $\overline{\rho}(\lambda_1) < \overline{\rho}(\lambda_2)$ and $R_\gamma(\lambda_1) \geq R_\gamma(\lambda_2)$, we obtain $\frac{\overline{\rho}(\lambda_1)}{R_\gamma(\lambda_1)} < \frac{\overline{\rho}(\lambda_2)}{R_\gamma(\lambda_2)}$, equivalent as $\varepsilon(\lambda_1) > \varepsilon(\lambda_2)$. ■

Theorem 5 points out that $\varepsilon(\lambda)$ decreases as λ increases. The next theorem provides a quantified measurement of $\varepsilon(\lambda)$.

Theorem 6: In a network where the nodes are randomly distributed in a Poisson point process with density λ and the optimal transmission radius $R_\gamma(\lambda)$ is used, defining $a = \lambda\pi R_\gamma^2(\lambda)$, $g_1(a) = \int_0^1 e^{a(x^2-1)}\mathrm{d}x$ and $g_2(a) = \int_0^1 e^{-\frac{1}{3}ax^2}\mathrm{d}x$,

$$g_1(a) \leq \varepsilon(\lambda) \leq g_2(a). \tag{10}$$

Proof: First, we define two relevant random variables that will be used in this proof. As depicted in Fig. 5(a), we define X_{max} as the distance between a node and its farthest neighbor within the transmission radius $R_\gamma(\lambda)$. In Fig. 5(b), we draw a sector at an arbitrary location o with radius $R_\gamma(\lambda)$ and central angle $\frac{2\pi}{3}$, and define X_{min} as the distance between o and the nearest node found in this sector.

Next, we prove $\varepsilon(\lambda) \geq g_1(a)$. As shown in Fig. 6, letting $\mathcal{P} = \{v_0, v_1, \cdots, v_{m-1}\}$ denote the relay path travelled by a packet from v_0 to reach z_m in m hops,

$$Z_m \leq \sum_{i=0}^{m-2} d_{v_iv_{i+1}} + d_{v_{m-1}z_m} \leq \sum_{i=0}^{m-1} X_{max,i} + R_\gamma(\lambda),$$

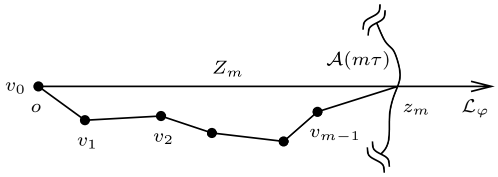


Fig. 6. Propagation distance Z_m in direction φ.

where $X_{max,i}$ is X_{max} taking place at v_i. Then,

$$\begin{aligned}\overline{\rho}(\lambda) = \lim_{m\to\infty}\frac{Z_m}{m} &\leq \lim_{m\to\infty}\frac{\sum_{i=0}^{m-1}X_{max,i}+R_\gamma(\lambda)}{m}\\ &= \lim_{m\to\infty}\frac{\sum_{i=0}^{m-1}X_{max,i}}{m} = E[X_{max}],\end{aligned}$$

since $X_{max,i}$ has i.i.d. probability distribution. We obtain $E[X_{max}]$ as follows. According to the Poisson distribution, with probability e^{-a} a node v_i has no neighbor, i.e., $X_{max} = 0$. With probability $1-e^{-a}$, v_i has at least one neighbor, i.e., $X_{max} > 0$. Given $0 < x \leq R_\gamma(\lambda)$,

$$\begin{aligned}\Pr[X_{max}\leq x|X_{max}>0] &= \frac{1}{1-e^{-a}}\sum_{k=1}^{\infty}e^{-a}\frac{a^k}{k!}\Big(\frac{\pi x^2}{\pi R_\gamma^2(\lambda)}\Big)^k\\ &= \frac{e^{-a}}{1-e^{-a}}\Big(e^{\frac{ax^2}{R_\gamma^2(\lambda)}}-1\Big).\end{aligned}$$

The conditional expectation is

$$\begin{aligned}E[X_{max}|X_{max}>0] &= \int_0^{R_\gamma(\lambda)} x\ \mathrm{d}\Pr[X_{max}\leq x|X_{max}>0]\\ &= \int_0^{R_\gamma(\lambda)}\frac{e^{-a}}{1-e^{-a}}\Big(\frac{2ax^2}{R_\gamma^2(\lambda)}\Big)e^{\frac{ax^2}{R_\gamma^2(\lambda)}}\mathrm{d}x\\ &= \frac{R_\gamma(\lambda)}{1-e^{-a}}\Big(1-\int_0^1 e^{a(x^2-1)}\mathrm{d}x\Big).\end{aligned}$$

The unconditional expectation is

$$\begin{aligned}E[X_{max}] &= e^{-a}\cdot 0 + (1-e^{-a})\cdot E[X_{max}|X_{max}>0]\\ &= R_\gamma(\lambda)\Big(1-\int_0^1 e^{a(x^2-1)}\mathrm{d}x\Big).\end{aligned}$$

Thus,

$$\varepsilon(\lambda) \geq \frac{R_\gamma(\lambda)-E[X_{max}]}{R_\gamma(\lambda)} = \int_0^1 e^{a(x^2-1)}\mathrm{d}x.$$

Finally, we prove $\varepsilon(\lambda) \leq g_2(a)$. As Fig. 7 illustrates, denote z_{m-1} as the farthest location that a packet has reached in direction φ after $m-1$ hops and $\mathcal{P} = \{v_0, v_1, \cdots, v_{m-2}\}$ as the relay path travelled by the packet to reach z_{m-1}. Draw a sector at z_{m-1} with radius $R_\gamma(\lambda)$ and central angle $\frac{2\pi}{3}$, as illustrated by the dash-line encompassed area in Fig. 7, where $\angle u_4 z_{m-1} u_6 = \angle u_5 z_{m-1} u_6 = \frac{\pi}{3}$. Note that for any node v' in this sector, $d_{v_{m-2}v'} \leq R_\gamma(\lambda)$, implying that v' must have received the packet by time $(m-1)\tau$ and forwarded the packet by time $m\tau$, i.e., $v' \in \widetilde{\mathcal{V}}(m\tau)$. Since z_m is the farthest location from o on $\mathcal{L}_\varphi$ covered by $\widetilde{\mathcal{V}}(m\tau)$, $d_{oz_m} \geq$

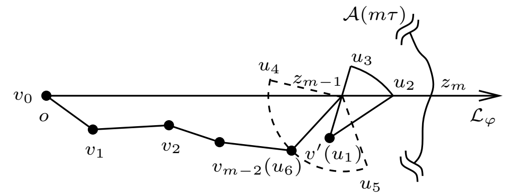


Fig. 7. The mth-hop propagation distance $\rho_m = Z_m - Z_{m-1}$ in direction φ, $d_{z_{m-1}u_4} = d_{z_{m-1}u_5} = d_{z_{m-1}u_6} = d_{u_1u_2} = d_{u_1u_3} = R_\gamma(\lambda)$, $\angle u_4 z_{m-1} u_6 = \angle u_5 z_{m-1} u_6 = \frac{\pi}{3}$.

d_{ou_2}, where u_2 is the farthest location reached by v' on $\mathcal{L}_\varphi$. So, $\rho_m = d_{z_{m-1}z_m} \geq d_{z_{m-1}u_2}$. By triangle inequality,

$$\rho_m \geq d_{z_{m-1}u_2} \geq d_{u_1u_2} - d_{u_1z_{m-1}} = R_\gamma(\lambda) - d_{u_1z_{m-1}}.$$

As the above inequality holds for all the v' in the sector,

$$\rho_m \geq \max_{\{v'\}}\{R_\gamma(\lambda) - d_{u_1z_{m-1}}\} = R_\gamma(\lambda) - X_{min},$$

where X_{min} is defined in Fig. 5(b). Replacing m with i,

$$\begin{aligned}\overline{\rho}(\lambda) = \lim_{m\to\infty}\frac{\sum_{i=1}^m \rho_i}{m} &\geq R_\gamma(\lambda) - \lim_{m\to\infty}\frac{\sum_{i=1}^m X_{min,i}}{m}\\ &= R_\gamma(\lambda) - E[X_{min}]\end{aligned}$$

where $X_{min,i}$ is X_{min} taking place at z_{i-1} and $X_{min,i}$ has i.i.d. probability distribution. Next, we compute $E[X_{min}]$. We know from the Poisson distribution that with probability $e^{-\frac{1}{3}a}$ there is no node in the sector, i.e., $X_{min} = R_\gamma(\lambda)$, and with probability $1-e^{-\frac{1}{3}a}$ there is at least one node in the sector, i.e., $X_{min} < R_\gamma(\lambda)$. Given $0 \leq x < R_\gamma(\lambda)$,

$$\begin{aligned}\Pr[X_{min}\leq x|X_{min}<R_\gamma(\lambda)] &= 1-\Pr[X_{min}>x|X_{min}<R_\gamma(\lambda)]\\ &= 1-\frac{e^{-\frac{1}{3}\lambda\pi x^2}-e^{-\frac{1}{3}a}}{1-e^{-\frac{1}{3}a}}.\end{aligned}$$

The conditional expectation is

$$\begin{aligned}E[X_{min}|X_{min}<R_\gamma(\lambda)] &= \int_0^{R_\gamma(\lambda)} x\ \mathrm{d}\Pr[X_{min}\leq x|X_{min}<R_\gamma(\lambda)]\\ &= \int_0^{R_\gamma(\lambda)}\frac{\frac{2}{3}\lambda\pi x^2 e^{-\frac{1}{3}\lambda\pi x^2}}{1-e^{-\frac{1}{3}a}}\\ &= \frac{R_\gamma(\lambda)}{1-e^{-\frac{1}{3}a}}\Big(\int_0^1 e^{-\frac{1}{3}ax^2}\mathrm{d}x - e^{-\frac{1}{3}a}\Big).\end{aligned}$$

The unconditional expectation is

$$\begin{aligned}E[X_{min}] &= e^{-\frac{1}{3}a}R_\gamma(\lambda)+(1-e^{-\frac{1}{3}a})E[X_{min}|X_{min}<R_\gamma(\lambda)]\\ &= R_\gamma(\lambda)\int_0^1 e^{-\frac{1}{3}ax^2}\mathrm{d}x.\end{aligned}$$

Thus,

$$\varepsilon(\lambda) \leq \frac{R_\gamma(\lambda)-(R_\gamma(\lambda)-E[X_{min}])}{R_\gamma(\lambda)} = \int_0^1 e^{-\frac{1}{3}ax^2}\mathrm{d}x.$$

■

Based on the result of Theorem 6, we are able to determine the asymptotic convergence rate of $\varepsilon(\lambda)$. In order to present this asymptotic rate, we prove the following lemma first.

Lemma 5: Define $h_1(b) = \int_0^1 k^{b(x^2-1)}\mathrm{d}x$ and $h_2(b) = \int_0^1 k^{-\frac{1}{3}bx^2}\mathrm{d}x$, where $k > 1$, $b > 0$. $\forall\ 0 < c < 1$ and $\epsilon > 0$, $h_1(b) > c^b$ and $h_2(b) < c^{b^{1-\epsilon}}$ as $b \to \infty$.

Proof: $\forall\ 0 < c < 1$, $\exists\ 0 < x_0 < 1$ s.t. $k^{x_0^2-1} > c$. Since $\int_0^1 (\frac{k^{x^2-1}}{c})^b \mathrm{d}x \geq \int_{x_0}^1 (\frac{k^{x^2-1}}{c})^b \mathrm{d}x \geq (1-x_0)(\frac{k^{x_0^2-1}}{c})^b \to \infty > 1$ as $b \to \infty$, $h_1(b) = \int_0^1 k^{b(x^2-1)}\mathrm{d}x > c^b$ as $b \to \infty$.

$\forall\ 0 < c < 1$, $\epsilon > 0$ and $x > 0$, as $b \to \infty$, $c^{b^{-\epsilon}} \to 1$ and $(\frac{k^{-\frac{1}{3}x^2}}{c^{b^{-\epsilon}}})^b \to 0$. Hence, $\int_0^1 (\frac{k^{-\frac{1}{3}x^2}}{c^{b^{-\epsilon}}})^b \mathrm{d}x \to 0 < 1$, which gives $h_2(b) = \int_0^1 k^{-\frac{1}{3}bx^2}\mathrm{d}x < c^{b^{1-\epsilon}}$ as $b \to \infty$. ■

The asymptotic convergence rates of $\varepsilon(\lambda)$ in the two noise models are then summarized in the next two theorems.

Theorem 7: In the ambience-dominant noise model, i) when $\lambda < \lambda_A$, there exist two constants c_1 and c_2 such that $c_1 \leq \varepsilon(\lambda) \leq c_2$; ii) when $\lambda > \lambda_A$, $\forall\ 0 < c < 1$ and $\epsilon > 0$, $c^\lambda < \varepsilon(\lambda) < c^{\lambda^{1-\epsilon}}$ as $\lambda \to \infty$.

Proof: When $\lambda < \lambda_A$, $R_\gamma(\lambda) = R_A\sqrt{\frac{\lambda_A}{\lambda}}$ and $a = \lambda_A \pi R_A^2$. By Theorem 6, letting $c_1 = g_1(\lambda_A \pi R_A^2)$ and $c_2 = g_2(\lambda_A \pi R_A^2)$, we have $c_1 \leq \varepsilon(\lambda) \leq c_2$.

When $\lambda > \lambda_A$, $R_\gamma(\lambda) = \left(\frac{P}{Ny(\alpha)}\right)^{\frac{1}{\alpha}}$ and $a = \lambda\pi(\frac{P}{Ny(\alpha)})^{\frac{2}{\alpha}}$. Choosing $k = e^{\pi(\frac{P}{Ny(\alpha)})^{\frac{2}{\alpha}}}$ and $b = \lambda$, by Theorem 6 and Lemma 5, $\forall\ 0 < c < 1$ and $\epsilon > 0$, as $\lambda \to \infty$,

$$\varepsilon(\lambda) \geq g_1(a) = h_1(\lambda) > c^\lambda,$$

and

$$\varepsilon(\lambda) \leq g_2(a) = h_2(\lambda) < c^{\lambda^{1-\epsilon}}.$$

■

Theorem 8: In the interference-dominant noise model, i) when $\lambda < \lambda_I$, there exist two constants c_1 and c_2 such that $c_1 \leq \varepsilon(\lambda) \leq c_2$; ii) when $\lambda > \lambda_I$, $\forall\ 0 < c < 1$ and $\epsilon > 0$, $c^{\lambda^{1-\frac{2}{\alpha}}} < \varepsilon(\lambda) < c^{\lambda^{(1-\frac{2}{\alpha})(1-\epsilon)}}$ as $\lambda \to \infty$.

Proof: When $\lambda < \lambda_I$, $R_\gamma(\lambda) = R_I\sqrt{\frac{\lambda_I}{\lambda}}$ and $a = \lambda_I \pi R_I^2$. By Theorem 6, letting $c_1 = g_1(\lambda_I \pi R_I^2)$ and $c_2 = g_2(\lambda_I \pi R_I^2)$, we have $c_1 \leq \varepsilon(\lambda) \leq c_2$.

When $\lambda > \lambda_I$, $R_\gamma(\lambda) = \left(\frac{P}{\lambda N(1)y(\alpha)}\right)^{\frac{1}{\alpha}}$ and $a = \lambda^{1-\frac{2}{\alpha}}\pi(\frac{P}{N(1)y(\alpha)})^{\frac{2}{\alpha}}$. Choosing $k = e^{\pi(\frac{P}{N(1)y(\alpha)})^{\frac{2}{\alpha}}}$ and $b = \lambda^{1-\frac{2}{\alpha}}$, by Theorem 6 and Lemma 5, $\forall\ 0 < c < 1$ and $\epsilon > 0$, as $\lambda \to \infty$,

$$\varepsilon(\lambda) \geq g_1(a) = h_1(\lambda^{1-\frac{2}{\alpha}}) > c^{\lambda^{1-\frac{2}{\alpha}}},$$

and

$$\varepsilon(\lambda) \leq g_2(a) = h_2(\lambda^{1-\frac{2}{\alpha}}) < c^{\lambda^{(1-\frac{2}{\alpha})(1-\epsilon)}}.$$

■

Theorems 7 and 8 reveal that in both noise models there is a threshold node density, below which $\varepsilon(\lambda)$ is bounded by constants (the constants are determined by the choice of parameter γ) and above which $\varepsilon(\lambda)$ converges to zero exponentially in the rates of $c^{\lambda^{1-\epsilon}}$ and $c^{\lambda^{(1-\frac{2}{\alpha})(1-\epsilon)}}$ respectively, where ϵ is an arbitrarily small positive real number.

VI. Conclusions

In this paper we have studied the information propagation speed in large wireless networks. We find that there is an upper bound, determined by the network parameters, on the information propagation speed. This upper bound is different for broadcast communications and unicast communications, but the two bounds converge in large networks. As a necessary condition for achieving this upper bound, all the nodes in the network must use an optimal transmission radius. We also reveal that, when a certain degree of satisfaction is required on delivering a packet to all the intended recipients, the speed upper bound is a function of the node density. In the ambience-dominant noise model, the bound is constant when the node density exceeds a threshold, while in the interference-dominant noise model, the bound decreases to zero as the node density grows to infinity. Finally, we prove that a packet propagates omnidirectionally in large random networks and the gap between its actual speed and the upper bound decreases as the node density increases. In both noise models, there exists a threshold node density, below which the gap is bounded by constants and above which the gap decreases to zero exponentially as node density increases to infinity. The work in this paper provides fundamental understanding of the fastest information delivery in large wireless networks.

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