On the Critical Phase Transition Time of Wireless Multi-hop Networks with Random Failures

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ABSTRACT

In this paper, we study the critical phase transition time of large-scale wireless multi-hop networks when the network topology experiences a partition due to increasing random node failures. We first define two new metrics, namely the *last connection time* and *first partition time*. The former is the last time that the network keeps a majority of surviving nodes connected in a single giant component; while the latter is the first time that the remaining surviving nodes are partitioned into multiple small components. Then we analyze the devolution process in a geometric random graph of n nodes based on percolation-theory connectivity and obtain the sufficient condition under which the graph is percolated. Based on the percolation condition, the last connection time and first partition time are found to be on the same order. Particularly, when the survival function of node lifetime is exponential, they are on the order of $\log(\log n)$; while if the survival function is Pareto, the order is $(\log n)^{1/\rho}$, where ρ is the shape parameter of Pareto distribution. Finally, simulation results confirm that the last connection time and first partition time serve as the lower and upper bounds of the critical phase transition time, respectively. Further, an interesting result is that the network with heavy-tailed survival functions is no more resilient to random failures than the network with light-tailed ones, in terms of critical phase transition time, if the expected node lifetimes are identical.

Categories and Subject Descriptors

G.3 [**Probability and Statistics**]: Stochastic processes; C.2.1 [**Computer-Communication Networks**]: Network Architecture and Design—*Network topology*

General Terms

Theory

Keywords

phase transition time, continuum percolation, wireless multi-hop networks, network devolution, random failures

1. INTRODUCTION

Wireless multi-hop networks are expected to operate in a decentralized and self-organized manner without fixed infrastructures, which makes them suitable to a variety of application scenarios, such as ad hoc battlefield deployments, environment surveillance/monitoring, and ubiquitous Internet access. A prerequisite for upper-layer application operations in such networks is that the underlying topologies should be connected. Thus, many works have been done to provide theoretical guidances on the critical transmission power/range, critical node degree, and critical density required for establishing (asymptotic) full connectivity [1–4]. For example, Xue and Kumar in [3] provided the critical degree for asymptotic connectivity to be $\Theta(\log n)$, [1] where n is the number of nodes in a network. Although these works have extended our understanding on wireless network connectivity and topology, a fundamental problem for large-scale deployments remains unsolved, that is, given a connected network in the presence of failures, *how long* can we expect the network sustains its connectivity before a dramatical change to its topology?

We notice that in the previous studies on connectivity, whether a network is connected or not is dependent on if there are isolated nodes. However, this full connectivity is impractical to achieve all the time in a large-scale wireless network and it does not provide a deeper insight to the entire topological devolution process. In a wireless network, node failures are common due to various reasons, such as power depletion, software or hardware malfunction, and arbitrary leave intrigued by end-users. When the number of node failures keeps increasing, the network inevitably devolves, which is characterized by a gradual shrinkage of the giant component size followed by an abrupt network connectivity breakdown. Understanding this process, especially the critical time when the network experiences topological transitions, is of importance in both theory and practice. For example, the critical phase transition time can be used as a

[1] We say $f(n) = O(g(n))$ if $\exists n_0 > 0$ and constant c_0 s.t. $f(n) \le c_0 g(n)$ $\forall n \ge n_0$. Similarly, we say $f(n) = o(g(n))$ if $f(n) < c_0 g(n)$. We say $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$ and $f(n) = \omega(g(n))$ if $g(n) = o(f(n))$. Finally, we say $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

theoretical estimation of network lifetime, a new metric for network survivability evaluation, or a temporal guidance for emergent network maintenance.

In this paper, we focus on the following question: *for a large-scale wireless multi-hop network in the presence of random failures, when does the network change its behavior from an almost connected phase to a fully partitioned phase?* Here a network is said to be *almost-connected* if there exists a giant component that is composed most of *surviving* nodes with high probability (w.h.p.); while a network is *fully-partitioned* if no such a giant component exists asymptotically almost surely (a.a.s.). [2] To tackle this problem, we couple a network devolution process with a continuum percolation process [5, 6] in a geometric random graph [7]. By using the concept of *percolation probability*, we first define two metrics, the *last connection time* and *first partition time*, to characterize the critical phase transition time. The former is the last time at which a network is almost connected and the latter is the first time at which a network is fully partitioned. Then we analyze the theoretical conditions under which a geometric random graph does (not) have a giant component of surviving nodes.

Through the analysis, we find that both the first partition time and last connection time are on the same order and depend on the network size n, initial density λ_0, transmission radius r_n, and survival function of node lifetime $S(t)$. In particular, when $S(t)$ is exponential, the order of the critical phase transition time is $\log(\log n)$; while if $S(t)$ is Pareto, this order is $(\log n)^{1/\rho}$, where ρ is the shape parameter of Pareto distribution. Finally, simulation results confirm that the phase transition period is bounded by the last connection time and first partition time from below and above, respectively. An interesting observation from our simulations is that a network with heavy-tailed $S(t)$ may even experience an earlier phase transition than a network with light-tailed $S(t)$ if expected node lifetimes are the same, which might be quite opposite to our intuition.

The rest of this paper is organized as follows. In Section 2, we introduce the models and percolation theory, and formulate the problem. In Section 3, we present our approach and main results, followed by potential applications. In Section 4, we provide detailed proofs for our analytical results. In Section 5, we use simulations to validate our analysis, followed by the conclusions in Section 6.

2. PROBLEM FORMULATION

In this section, we first introduce the models and assumptions, then define two new metrics by using preliminaries from percolation theory, and formulate the problem last.

2.1 Models and Assumptions

Network model: In this paper, a wireless multi-hop network is modeled by a *geometric random graph* [7] and denoted by $G(\mathcal{X}_n, r_n)$, where $\mathcal{X}_n \triangleq \{X_n, n \in \mathbb{N}\}$ denotes the node set and r_n denotes the node transmission radius. In this model, $X_1, X_2, \cdots, X_n$ denote the random locations of nodes in $\mathbb{R}^2$ and they are independently and identically distributed (i.i.d.) in a square with area A according to a two dimensional *uniform distribution*. The transmission radius r_n is assumed to be identical for all nodes so that undirected links exist in the graph if $\|x_i - x_j\| \leq r_n$ holds for node pairs (x_i, x_j). When $n \to \infty$ and $A \to \infty$ with $\frac{n}{A} = \lambda$ fixed, $\mathcal{X}_n$ can be presented by a homogeneous Poisson point process $\mathcal{H}_\lambda$ with density λ, which results in an (infinite) geometric random graph $G(\mathcal{H}_\lambda, r_n)$ [7, 8].

Nevertheless, in a real wireless multi-hop network, nodes are actually distributed in a bounded domain. Although we are interested in scaling laws on the critical phase transition time for large-scale networks (in the sense of large n), it is natural to study *finite* random geometric graphs. To define precisely, let $B(s_n)$ denote a continuum box with side length $s_n = \sqrt{n/\lambda_0} > 0$ and let $\mathcal{H}_{\lambda_0,s_n}$ be the restriction of a Poisson process $\mathcal{H}_{\lambda_0}$ of density λ_0 to the box $B(s_n)$, i.e., $\mathcal{H}_{\lambda_0,s_n} \triangleq \mathcal{H}_{\lambda_0} \cap B(s_n)$. Then the network studied in this work can be formally modeled by a random geometric graph $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ in which two points of $\mathcal{H}_{\lambda_0,s_n}$ are adjacent if their distance is at most r_n, as aforementioned.

REMARK 1. In our network model, the node density λ_0 is fixed to a given constant and the number of nodes n is increased (to infinity) by increasing the deployment region $B(s_n)$ (to infinity). This is also called the *extended network model* [9, 10]. Notice that it has been proved that the critical node degree for connectivity is $\Theta(\log n)$ [3, 8]. Thus we need to scale r_n with n to obtain an initially connected network with high probability.

Random failure model: In order to describe the impact of node failures on the devolution of wireless multi-hop networks, we introduce a random failure model to extend our network model $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ defined above. In this failure model, each node is either *surviving* or *failed* at any time and a failed node does not recover back to surviving state. Let T_i $(1 \leq i \leq n)$ denote the lifetime of node i before it is failed, then $T_1, \cdots, T_n$ are random variables, which are assumed i.i.d.. The *complementary* cumulative distribution function (c.d.f.) of the node lifetime is called the *survival function*, denoted by $S(t) \triangleq Pr(T_i > t)$. The survival function $S(t)$ actually serves as the probability that a node is surviving at time t, which will be used extensively in our succeeding analysis.

Interference model: An intrinsic feature of a wireless multi-hop network is that all communications share open radio channels, which makes signal interference as an unnegligible factor in studying network connectivity [11]. In a widely used interference model (see [9, 11–13] for example), the condition for a successful transmission from node i to node j is that the *signal-to-interference-with-noise-ratio (SINR)* should be above a certain threshold, say β. Formally, the condition can be written as $\frac{P_r(i,j)}{N_0 + \gamma \sum_{k \neq i,j} P_r(k,j)} > \beta$, where $P_r(x, j)$ is the signal of x received by j, N_0 is the power of the thermal background noise, and γ is the inverse of the processing gain of the system, called *orthogonality factor* [11]. This interference model indicates that there is an upper limit on the number of adjacent nodes (degree), which was proved to be $1 + 1/(\gamma\beta)$ in [11]. This upper limit on the node degree is of our interest and we call it the *degree bound*, denoted by $K \triangleq 1 + 1/(\gamma\beta)$, in the following context.

2.2 Preliminary of Percolation

To further explain how the continuum percolation process on a geometric random graph is related to the topo-

[2] An event A_n occurs w.h.p. or a.a.s. if $Pr(A_n) \to 1$ as $n \to \infty$. We use "w.h.p." and "a.a.s." interchangeably in this paper. For more details, please refer to [8] (Chapter 8).

logical devolution process of a large-scale wireless multi-hop network, we introduce some percolation terminologies first. From graph theory, we know that a *component* of a graph G is a *maximal* connected subgraph of G. With this definition, the *percolation probability* is defined as follows [5, 7, 8]

Definition 1. *Let $\mathcal{C}_\mathbf{0}$ be the component of a graph $G(\mathcal{H}_\lambda, r_n)$ containing the origin of $\mathbb{R}^2$, $|\mathcal{C}_\mathbf{0}|$ be the number of points in $\mathcal{C}_\mathbf{0}$, then the* percolation probability*, denoted by $p_\infty(\lambda)$ for density λ, is the probability that $\mathcal{C}_\mathbf{0}$ contains infinite points, i.e., $p_\infty(\lambda) \triangleq Pr(|\mathcal{C}_\mathbf{0}| = \infty)$.*

A fundamental result of continuum percolation is that there exists a *critical density* λ_c defined by $\lambda_c \triangleq \inf\{\lambda > 0 : p_\infty(\lambda) > 0\}$ so that: if $\lambda > \lambda_c$, the graph $G(\mathcal{H}_\lambda, r_n)$ is in the *super-critical* phase and $p_\infty(\lambda) > 0$; while if $\lambda < \lambda_c$, $G(\mathcal{H}_\lambda, r_n)$ is said to be *sub-critical* and $p_\infty(\lambda) = 0$ [5, 6]. When the graph is super-critical, $\mathcal{C}_\mathbf{0}$ is normally called the *giant component* since it contains most of (and indeed, an infinite number of) the nodes in a network.

Although we are interested in the percolation in large-scale, finite wireless multi-hop networks, the above percolation probability definition can be applied to our network model $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ as well. By using similar notations, we can define the percolation probability with large n for $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ as $p_\infty(\lambda) \triangleq Pr(|\mathcal{C}_\mathbf{0}| \geq n)$, where n is the expected number of points in the continuum box $B(s_n)$. Once again, as $n \to \infty$, our definition converges to the original definition. Moreover, the critical density definition also applies to our network model and there exists a unique giant component containing $\Theta(n)$ points in $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ almost surely as $n \to \infty$ if $\lambda_0 > \lambda_c$.

Given a large-scale wireless network with each node associated with a survival function $S(t)$, according to the *Thinning theorem* (*Theorem 9.15* [7]), the point process of surviving nodes is also a Poisson process with density function $\lambda_1(t) \triangleq \lambda_0 S(t)$. As time goes, although more and more failures are present, as long as $\lambda_1(t)$ is high enough, most of surviving nodes are still in a giant component. Once $\lambda_1(t)$ drops below a certain point λ_c, the connectivity among the surviving nodes breakdowns quickly and no giant component exists any more. Thus, the percolation process is a natural analogy to the devolution process aforementioned.

We notice that the percolation theory, especially the continuum percolation model, has been used to analyze the connectivity, capacity, and latency of wireless networks recently [11, 14–16]. For example, Kong et al recently in [16] modeled an energy scheduling mechanism by a *degree-dependent* dynamic site percolation process on random geometric graphs. The above works demonstrate the applications of the percolation theory in wireless networks; while none of them ever addresses our NPT-problem, defined right next.

2.3 Problem Formulation

In order to understand the critical phase transition time during network devolution, we first define *almost connected* and *fully partitioned* networks as follows:

Definition 2. *Let $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ be a geometric random graph, in which every point is associated with the same survival function $S(t)$. Let $\lambda_1(t) \triangleq \lambda_0 S(t)$, then the network represented by $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ is said to be almost connected if $p_\infty(\lambda_1(t)) > 0$, and fully partitioned if $p_\infty(\lambda_1(t)) = 0$, where $p_\infty(\cdot)$ is the percolation probability.*

Next we define two new metrics called the *last connection time* and *first partition time*.

Definition 3. *With the same conditions given in* Definition 2*, the* last connection time *is defined by*

$$t_c(n) \triangleq \sup\{t > 0 : p_\infty(\lambda_1(t)) > 0\}, \tag{1}$$

where $\lambda_1(t) \triangleq \lambda_0 S(t)$. The first partition time *is defined by*

$$t_p(n) \triangleq \inf\{t > 0 : p_\infty(\lambda_1(t)) = 0\}. \tag{2}$$

Definition 4. *The critical phase transition time, denoted by $\mathcal{T}_C$, is the critical time point above which $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ is disconnected a.a.s. (sub-critical) and below which $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ is connected a.a.s., (super-critical), that is*

$$\lim_{n\to\infty} Pr(G \text{ is connected}) = \begin{cases} 1, & \text{if } t < \mathcal{T}_C, \\ 0, & \text{if } t > \mathcal{T}_C. \end{cases} \tag{3}$$

The exact value of T_c is unknown, but it is expected to be bounded by $t_c(n)$ and $t_p(n)$ from below and above, respectively, based on our definitions on $t_c(n)$ and $t_p(n)$.

Now we formulate the problem addressed in this paper as the *Network Partition Time (NPT)* problem.

Definition 5. **(NPT problem)***: For a large-scale wireless multi-hop network represented by a geometric random graph $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$, every node is assumed to be independently associated with survival function $S(t)$. Given the network is fully connected at initial time $t = 0$, find out*

1. *the relations among n, λ_0, r_n, and $S(t)$ that would be sufficient to guarantee that the network is almost connected or fully partitioned, respectively;*
2. *the upper limit of $t_c(n)$ and the lower limit of $t_p(n)$, such that the critical phase transition time $\mathcal{T}_C$ can be bounded by these limits.*

The results of this problem reveal that when time $t < t_c(n)$, the network is guaranteed to be almost connected (super-critical); while $t > t_p(n)$ will be sufficient for the network to be fully partitioned (sub-critical). We expect the bounds of critical phase transition time (i.e., $t_c(n)$, $t_p(n)$) to be tight so that the phase transition is sharp and the *period of phase transition* (i.e., the gap between $t_c(n)$ and $t_p(n)$) converges to 0 as fast as possible. Nevertheless, in practical, a longer period of phase transition might be preferable to provide a smooth degradation of connectivity.

3. RESULTS AND APPLICATIONS

Intuitively, a trivial solution to the NPT problem tells that: when $S(t) > \lambda_c/\lambda_0$, most of surviving nodes are connected in the giant component a.a.s.; otherwise, no surviving node belongs to a component that is significantly larger than any other components. If we know λ_c, given $S(t)$ with any specific distribution, we can find $t_c(n)$ and $t_p(n)$ directly. However, the exact value for λ_c is unknown, although some numeric bounds were obtained from rigorous mathematical proofs with a wide range, e.g., $0.696 < \lambda_c < 3.372$ [5], or from computer experiments with little theoretical justification [17]. Therefore, the NPT problem is a quite challenging problem in the continuous domain.

Our approach takes following procedures. We first map the percolation process defined on the continuous plane onto a discrete lattice, whose edges are declared *open* if certain properties of the Poisson process in their vicinities are met (*closed* otherwise). In the discrete lattice, we then investigate the condition when infinite open paths (composed of open edges) exist with positive probability. With a careful definition on the open edge in the lattice, a reverse mapping can be carried out back to the continuous plane so that infinite open paths on the discrete plane indicate connected components on the continuous plane. Finally, we obtain the continuum percolation conditions, which enable us to derive the bounds of critical phase transition time, i.e., $t_c(n)$ and $t_p(n)$, for given survival functions.

3.1 Contributions

We summarize our main results as follows. First, following theorems provide the critical conditions for a network to be almost connected and fully partitioned, respectively, thus solving the first part of the NPT problem.

THEOREM 1. *Given a graph $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$, assume $\mu_0 = \lambda_0 \pi r_n^2 = \Theta(\log n)$ and the degree bound $K = (1+\epsilon_n)\mu_0$ where ϵ_n is an arbitrary increasing function of n. There exists a positive constant c_ϵ, such that if the survival function $S(t)$ satisfies,*

$$S(t) > \frac{\sqrt{5}(\ln 18 - \ln(1 - 15\phi_n))}{c_\epsilon r_n \sqrt{\lambda_0 \ln n}}, \qquad (4)$$

where $\phi_n = 1 - \exp(-\frac{2c_\epsilon^2 \ln n}{\epsilon_n^2 \mu_0})$, then $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ is in the super-critical phase.

THEOREM 2. *Given a graph $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$, assume $\mu_0 = \Theta(\log n)$ and $K = (1 + \epsilon_n)\mu_0$. There exists a positive constant c_ϵ, such that if the survival function $S(t)$ satisfies:*

$$S(t) < \frac{\ln \sqrt{3\psi_K} - \ln(\sqrt{3\psi_K} - 1)}{c_\epsilon r_n \sqrt{\lambda_0 \ln n}}, \qquad (5)$$

where $\psi_K = \frac{\Gamma(K,\mu_0)}{(K-1)!}$ and $\Gamma(x, y)$ is the incomplete Gamma function, then $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ is in the sub-critical phase.

REMARK 2. The assumptions on μ_0 and K are needed to achieve an initial fully connectivity, which is the condition of the NPT problem. Further, they also guarantee $\psi_K > \frac{1}{3}$ such that $\ln(\sqrt{3\psi_K} - 1)$ is a real number in (5). In fact, the format of K indicates that the impact of interference has to be sufficiently small so that the degree bound could be large enough to support the percolation as $n \to \infty$, which is in accordance with the result proved in [11]. Detailed explanations are given in Section 4.3.

Next, following corollaries answer the second part of the NPT problem, providing the theoretical bounds on the critical phase transition time.

COROLLARY 1. (Limits of $t_c(n)$ and $t_p(n)$ with light-tailed $S(t)$): *Assume that $S(t) = e^{-\alpha t}$ (exponential), where the positive $1/\alpha$ represents the mean lifetime of a node, then the upper limit of last connection time $t_c(n)$ is,*

$$t_c(n) = \frac{1}{\alpha}\ln(\ln n) + c_1 \ \sim \Theta(\log(\log n)), \qquad (6)$$

where $c_1 = \frac{1}{\alpha}(\ln(c_\epsilon\sqrt{\frac{c}{\pi}}) - \ln(\sqrt{5}\ln\frac{18}{1-15\phi_n}))$ and $c \triangleq \frac{\mu_0}{\ln n}$. The lower limit of first partition time $t_p(n)$ is,

$$t_p(n) = \frac{1}{\alpha}\ln(\ln n) + c_2 \ \sim \Theta(\log(\log n)), \qquad (7)$$

where $c_2 = \frac{1}{\alpha}(\ln(c_\epsilon\sqrt{\frac{c}{\pi}}) - \ln(\ln\frac{\sqrt{3\psi_K}}{\sqrt{3\psi_K}-1}))$.

COROLLARY 2. (Limits of $t_c(n)$ and $t_p(n)$ with heavy-tailed $S(t)$): *Assume that $S(t) = (t/\eta)^{-\rho}$ (heavy-tailed Pareto, $\rho > 1$) with mean $\frac{\eta\rho}{\rho-1}$, then the upper limit of $t_c(n)$ is,*

$$t_c(n) = c_3(\ln n)^{1/\rho} \ \sim \Theta((\log n)^{1/\rho}), \qquad (8)$$

where $c_3 = \eta(\frac{c_\epsilon\sqrt{c/\pi}}{\sqrt{5}(\ln 18 - \ln(1-15\phi_n))})^{1/\rho}$ and $c \triangleq \frac{\mu_0}{\ln n}$. The lower limit of $t_p(n)$ is

$$t_p(n) = c_4(\ln n)^{1/\rho} \ \sim \Theta((\log n)^{1/\rho}), \qquad (9)$$

where $c_4 = \eta(\frac{c_\epsilon\sqrt{c/\pi}}{\ln(\sqrt{3\psi_K}) - \ln(\sqrt{3\psi_K}-1)})^{1/\rho}$.

COROLLARY 3. *The critical phase transition time, $\mathcal{T}_C$, is bounded by the last connection time and first partition time, i.e., $t_c(n) \leq \mathcal{T}_C \leq t_p(n)$.*

REMARK 3. A premise used in above theorems is $\mu_0 = c\ln n = \Theta(\log n)$, where c is some constant such that the network is fully connected initially. In particular, Xue and Kumar proved in [3] that $5.1774\log n$ is required for a.a.s. connectivity and this threshold was further improved by Balister et al in [18] to $0.5139\log n$ (also see [8]). However, in our simulations, we find that $0.5139\log n$ is far less sufficient to achieve an initially connected random topology and actually $5.1774\log n$ is a "good" threshold for connectivity.

REMARK 4. From reliability engineering, we know that many lifetime distributions (e.g., exponential, log-normal, Pareto, Weibull) are either light-tailed or heavy-tailed according to the decay speed of their tails. Since the exponential distribution is the only distribution to have a constant failure rate and applies naturally to model memoryless lifetime, it is used to represent light-tailed survival functions; while the Pareto distribution is used to represent heavy-tailed survival functions when node lifetimes are power law distributed or have very large variance.

3.2 Applications

Besides the theoretical importance of our findings, our results can be used practically not only in the initial deployment and dynamic reconfiguration of a wireless multi-hop network, but also as a benchmark in evaluating other protocol designs. Here are some examples.

- In the initial network deployment, an appropriate value for the number of nodes can be decided to guarantee an almost connected network lasting for an expected time, if transmission radius r_n and statistical distribution of node lifetime $S(t)$ are known.

- In sensor networks, nodes are very vulnerable to multiple failures, affecting the communication connectivity and in turn impairing the sensing coverage. As pointed out in [19], redeploying additional nodes is necessary to replace failed sensors so that a connected network topology can be maintained. However, for many unattended outdoor sensor network applications, it is usually inefficient and costly to replace failed sensors one by one. Our results provide network designers a guideline on the optimal time that the redeployment of additional sensors should be carried out.

- There are many power management and topology control algorithms, such as [20, 21], proposed to minimize the energy consumption in order to maintain network connectivity as well as prolong network lifetime. Our results, without involving any specific power-saving schemes, can be used as a benchmark for the expected network lifetime to evaluate the performance of existing and future algorithm designs whenever the network lifetime is of interest.

4. TRANSITION TIME ANALYSIS

In this section, we demonstrate how to obtain the results given in Section 3. Our approach is to map the continuous and discrete planes first, and define *open* (*closed*) edges on the discrete plane. Then we find percolation conditions on the continuous plane relating to the survival function $S(t)$. At last, we derive the limits on the first partition time and last connection time by specifying $S(t)$ with light-tailed and heavy-tailed distributions.

4.1 Mapping and Open Edge Definition

We begin by constructing a square lattice, denoted by $\mathcal{L}$ over the plane, with edge length d. Let $\mathcal{L}'$ be the dual lattice of $\mathcal{L}$, constructed by putting a vertex in the center of every face (square) of $\mathcal{L}$, and an edge across every edge of $\mathcal{L}$. Since $\mathcal{L}$ is a square lattice, $\mathcal{L}'$ is simply the same lattice shifted by $d/2$ horizontally and vertically, as depicted in Figure 1. Without losing generality, we further set the origin $\mathbf{0}$ of the plane at a vertex of $\mathcal{L}$.

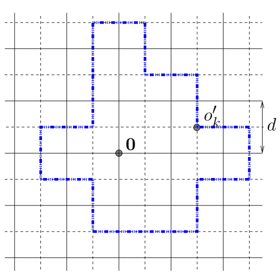


Figure 1: The lattice $\mathcal{L}$ (solid), its dual $\mathcal{L}'$ (dashed), and a circuit (bold dashed) on $\mathcal{L}'$.

To define an open edge, we use the *crossing* property [5–7] to describe the connection among multiple points.

DEFINITION 6. *In a 2-D plane, let $X_v = (x_v, y_v)$ be the position of a point v, where x_v and y_v denote v's x-coordinate and y-coordinate, respectively. For a 2-D box $B \triangleq [0, l_1] \times [0, l_2]$, if there exist a series of points $v_1, v_2, \cdots, v_m$ within B such that $\forall 1 \le i < j \le m$ $x_{v_i} < x_{v_j}$, $0 < x_{v_1} < r$, $l_1 - x_{v_m} < r$, and $\|X_{v_{i+1}} - X_{v_i}\| \le r$, then B is said to be crossed by a component from left to right or have an* LR-crossing. *If the conditions above are satisfied in B when x and l_1 are substituted by y and l_2, respectively, B is said to be crossed from top to bottom or have a* TB-crossing.

For every horizontal edge a of $\mathcal{L}$, let (x_a, y_a) be the coordinates of the point in the center of a and the rectangle B_a be the vicinity of a. We introduce an event E_a that occurs as a result of following three events,

1. $LR_a \triangleq \{$ there is an LR-crossing in the rectangle $B_a \triangleq [x_a - d, x_a + d] \times [y_a - \frac{d}{2}, y_a + \frac{d}{2}]\}$,

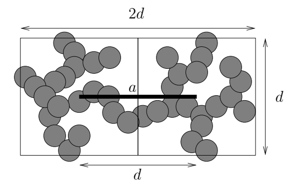


Figure 2: A horizontal edge a that fulfills the LR-crossing and TB-crossing.

2. $TB_a^L \triangleq \{$ there is a TB-crossing in the "left" square $B_a^L \triangleq [x_a - d, x_a] \times [y_a - \frac{d}{2}, y_a + \frac{d}{2}]\}$, and
3. $TB_a^R \triangleq \{$ there is a TB-crossing in the "right" square $B_a^R \triangleq [x_a, x_a + d] \times [y_a - \frac{d}{2}, y_a + \frac{d}{2}]\}$.

The occurrence of the event E_a is illustrated in Figure 2 where balls represent Poisson points. Note E_a can be defined similarly for vertical edges by exchanging the notations of x_a and y_a in the conditions above.

As aforementioned in Section 2.1, due to signal interference, the number of neighbors of any node should be upper bounded by the degree bound K (see *Theorem 1* in [11] for details). Thus we define another event E'_a that occurs in the rectangle B_a if and only if each point in B_a has no more than K neighbors. The occurrence of E'_a guarantees that no point in B_a is isolated from others because of signal interferences. Let D_v be the number of neighbors of a point v, E'_a is formally represented by

$$E'_a \triangleq \{D_v < K, \ \forall\ v \text{ and } X_v \in B_a\}. \tag{10}$$

With the two events described above, we can rigorously define *open* edges as follows.

DEFINITION 7. *In the lattice $\mathcal{L}$, an edge a is said to be* open *if and only if both events E_a and E'_a occur in its associated rectangle B_a; and* closed *otherwise. In the dual lattice $\mathcal{L}'$, an edge a' is open if there is an open edge of $\mathcal{L}$ crossing a'; otherwise, a' is closed.*

This open edge definition bridges the discrete and continuous planes in that a cluster comprised of *adjacent* [3] open edges in $\mathcal{L}$ correspond to a unique component (comprised of surviving nodes) in $G(\mathcal{H}_{\lambda_0,s}, r)$. This rationale guarantees the validity of our mapping approach, which is summarized in the following result.

LEMMA 1. *Given the mapping and open edge defined above, if there exists an infinite open edge cluster in $\mathcal{L}$, then there exists a giant component in $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$.*

PROOF. See Appendix 8.1. $\square$

Lemma 1 implies that if the lattice $\mathcal{L}$ is percolated then the continuum percolation also occurs in graph $G(\mathcal{H}_{\lambda_0,s_n}, r)$; and vice versa. Thus, the percolation conditions obtained in $\mathcal{L}$ can be applied to continuum percolation, which will be explained in detail next.

4.2 Percolation Condition in Lattice

In discrete percolation theory, the open or close state of every edge (or vertex) is independent from others. The

[3] Two edges are adjacent if they are incident to the same vertex in the lattice.

percolation probability is dependent on a certain *critical probability*, defined by $p_c = \sup\{p : p_\infty(p) = 0\}$, where p is the probability of any edge being open. It was proved in [5, 6] that non-trivial upper and lower bounds for p_c are $\frac{1}{3} \leq p_c \leq \frac{2}{3}$ in square lattices, and more precisely, $p_c = \frac{1}{2}$.

In our discrete lattice mapping, the state of an edge depends on, however, how the Poisson points surrounding the edge are connected, which implies that adjacent edges are not independent. Therefore, we cannot directly use the theoretical bounds given in discrete percolation theory and we need to find out alternative percolation conditions for our mapping. The percolation condition on $\mathcal{L}$ is based on the following fact.

LEMMA 2. *Given a lattice $\mathcal{L}$ containing the origin $\mathbf{0}$, let $\sigma(m)$ be the number of paths with length m (i.e., comprising m edges) that start at $\mathbf{0}$, then $\sigma(m) \leq 4 \cdot 3^{m-1}$. Further, let $\rho(m)$ be the number of circuits* [4] *in the dual lattice $\mathcal{L}'$ with length m and containing $\mathbf{0}$ in their interiors, then $\rho(m) \leq 2 \cdot (m-2) \cdot 3^{m-2}$.*

PROOF. See Appendix 8.2. □

By using the fact above, we have

LEMMA 3. *For the given lattice $\mathcal{L}$ constructed above, let p be the probability that an edge is open, if $p > \frac{14}{15}$, then there exists an infinite open edge cluster in $\mathcal{L}$, i.e., $p_\infty > 0$; if $p < \frac{1}{9}$, then there is no percolation in $\mathcal{L}$, i.e., $p_\infty = 0$.*

PROOF. See Appendix 8.3. □

Lemma 3 provides us a useful tool to study the percolation on the continuous plane. For example, if we can find an upper bound of p such that $p < \frac{1}{9}$, we are able to derive the condition for non-percolation on the continuum space.

In the following context, the concept of *increasing (decreasing) event* and two powerful inequalities will be frequently used. We introduce them as follows.

DEFINITION 8. *For two geometric random graphs $\mathcal{G}$ and $\mathcal{G}'$, a partial ordering $\preceq$ is defined as $\mathcal{G} \preceq \mathcal{G}'$ if and only if $\mathcal{G}'$ can be induced from $\mathcal{G}$ by adding more (Poisson) points. Then an event A is said to be* increasing *(*decreasing*) if $\forall\ \mathcal{G} \preceq \mathcal{G}'$ and $1_A(\mathcal{G}) \leq 1_A(\mathcal{G}')$ $(1_A(\mathcal{G}) \geq 1_A(\mathcal{G}'))$, where 1_A is the indicator function of event A.*

LEMMA 4. (Reimer's inequality [6]) *For two events A_1 and A_2, if A_1 is increasing and A_2 is decreasing, then $Pr(A_1 \cap A_2) \leq Pr(A_1)Pr(A_2)$.*

LEMMA 5. (FKG's inequality [5,6]) *If two events A_1 and A_2 are both increasing or decreasing, then $Pr(A_1 \cap A_2) \geq Pr(A_1)Pr(A_2)$.*

Recall that we have defined an edge a open in *Definition 7* by using two events E_a and E'_a, where E_a occurs if there exist an LR-crossing and two TB-crossings in the rectangle B_a, and E'_a occurs if every point in B_a has no more than K neighbors. Since the more points are in B_a, the more likely E_a occurs while the less likely E'_a occurs, based on *Definition 8*, we know that E_a is increasing and E'_a is decreasing. Thus, by Reimer's inequality, we have an upper bound of p as

$$p = Pr(E_a \cap E'_a) \leq Pr(E_a)Pr(E'_a). \tag{11}$$

[4] A circuit in a lattice is a path that two ends are the same vertex without repeated intermediate vertices.

Noticing that the two events E_a and E'_a are not independent, we have a lower bound of p as

$$\begin{aligned} p &= Pr(E_a) + Pr(E'_a) - Pr(E_a \cup E'_a) \\ &\geq Pr(E_a) + Pr(E'_a) - 1. \end{aligned} \tag{12}$$

In the next two subsections, we prove the two main theorems by further investigating (11) and (12), respectively.

4.3 Condition for Continuum Percolation

In this subsection, we investigate the sufficient condition for the percolation in the continuous domain and prove *Theorem 1.* More specifically, our target is to find out the condition for $Pr(E_a) + Pr(E'_a) - 1 > \frac{14}{15}$ because of *Lemma 3.* In the following, we will study the lower bounds of $Pr(E_a)$ and $Pr(E'_a)$ first, and define the proper order of the side length d of the lattice next, then prove our first main theorem last.

The Lower Bound of $Pr(E_a)$: Intuitively, the more points are in a box, the more likely there is a component in the box. Thus, LR_a, TB_a^L, and TB_a^R are all increasing events. By FKG's inequality, $Pr(E_a)$ is lower bounded by

$$\begin{aligned} Pr(E_a) &= Pr(LR_a \cap TB_a^L \cap TB_a^R) \\ &\geq Pr(LR_a)Pr(TB_a^L)Pr(TB_a^R). \end{aligned} \tag{13}$$

Note that the points are distributed homogeneously with random failures, we have $Pr(TB_a^L) = Pr(TB_a^R)$ in (13). In addition, for squares with the same size, an LR-crossing and a TB-crossing occur with the same probability. By *Lemma 11.73* and *11.75* in [6], let $Pr(TB_a^L) = Pr(TB_a^R) = \tau$, we have $Pr(LR_a) \geq \tau(1 - \sqrt{1-\tau})^6$, which yields

$$\begin{aligned} Pr(E_a) &\geq \tau^3(1 - \sqrt{1-\tau})^6 \\ &= 1 - 6\sqrt{1-\tau} + o(\sqrt{1-\tau}). \end{aligned} \tag{14}$$

To proceed, we need the following known result.

LEMMA 6. (Lemma 10.5 [7]) *For a geometric random graph $G(\mathcal{H}_\lambda, r_n)$ in 2-D space, if $\lambda > \lambda_c$, then there exist $c > 0$ and $d_1 > 0$ such that $1 - Pr(LR_a) \leq \exp(-cd)$ for all $d \geq d_1$*

This implies that the probability that a square box $[0, d]^2$ has an LR-crossing approaches 1 when d is sufficiently large, given that the graph is super-critical. Based on this result, from (14) we have

LEMMA 7. *Assume $\lambda_0 S(t) > \lambda_c$, the probability that the event E_a occurs is lower bounded by*

$$Pr(E_a) \geq 1 - \frac{6}{5}\exp\left(-\frac{\sqrt{5}}{5}\lambda_0 r_n d S(t)\right). \tag{15}$$

PROOF. See Appendix 8.4. □

The Lower Bound of $Pr(E'_a)$: Next, we investigate the lower bound of $Pr(E'_a)$. Let μ_0 denote the expected node degree, it is known that each point has an approximately Poisson degree distribution [7, 8, 22, 23] as n is sufficiently large. Thus, the expected degree μ_0 can be given by

$$\mu_0 = \lim_{n\to\infty} \frac{(n-1)\lambda_0 \pi r_n^2}{n} = \lambda_0 \pi r_n^2. \tag{16}$$

Then we have the following observation.

LEMMA 8. *Let $k = \frac{K-\mu_0}{\sqrt{\mu_0}} > 0$, where K is the degree bound and μ_0 is given by* (16)*, then the probability that the event E'_a occurs is lower bounded by*

$$Pr(E'_a) \geq \left(1 - \frac{1}{1+k^2}\right)^{2\lambda_0 d^2}. \tag{17}$$

PROOF. See Appendix 8.5. □

The Tight Bound of d: It is noticed that the lower bounds of $Pr(E_a)$ in (15) and $Pr(E'_a)$ in (17) are functions of the edge length d in the lattice $\mathcal{L}$. In order to make our results independent from the mapping scheme with specific value of d, we need to define d properly by using basic network parameters, such as n, λ_0, and r_n. However, defining the value of d precisely or even confining d to a proper order is a challenging problem. Here we have:

PROPOSITION 1. *Given the mapping between the graph $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$ and lattice $\mathcal{L}$ described in Section 4.1, the edge length d has a tight bound as $d = \Theta(\sqrt{\log n})$.*

PROOF (SKETCH). We first look at the upper bound of d. An extreme case is $d = \Theta(s_n)$. In this case, the problem of finding the probability of an open edge will be equivalent to the problem of finding the probability that there exists a giant component in the graph. Thus, s_n is too loose to bound d from above and d should be smaller than s_n, i.e., $d = o(s_n)$.

Recall that we assume that the network is fully connected initially at time $t = 0$ in our problem formulation (*Definition 5*), then it is reasonable to assume that the degree bound K should be greater than the expected degree μ_0, i.e., $K = \Omega(\mu_0)$; otherwise, all points with more than μ_0 neighbors may be isolated "logically" already even without any failures. Further, as we mentioned in Section 2.1, in order to achieve the initial connectivity w.h.p., each point should be connected to $\Theta(\log n)$ neighbors, which implies that μ_0 is also on the order of $\log n$ so that the network is connected a.a.s.. Thus, $K = \Omega(\mu_0)$ goes to infinity as $n \to \infty$, and consequently, $k \triangleq \frac{K-\mu_0}{\sqrt{\mu_0}} = \Omega(\sqrt{\log n})$. From (17), we have

$$\lim_{n\to\infty} Pr(E'_a) \geq \lim_{n\to\infty} \left(1 - \frac{1}{k^2}\right)^{2\lambda_0 d^2} = \exp\left(-\frac{2\lambda_0 d^2}{k^2}\right).$$

The above equation tells that in order to guarantee the event E'_a to occur w.h.p., d needs to be less than k, i.e., $d = o(k)$; otherwise, it is quite impossible to achieve $p = Pr(E_a \cap E'_a) > \frac{14}{15}$ when n goes to large, even if $Pr(E_a) = 1$. Therefore, we have $\sqrt{\log n}$ as a (tighter) asymptotic upper bound for d, i.e., $d = O(\sqrt{\log n})$.

Next we look at the lower bound of d. It is also easy to know that d should be at least greater than r_n; otherwise event E_a cannot be guaranteed to occur w.h.p., according to *Lemma 6*. Since $\mu_0 = \lambda_0 \pi r^2 = \Theta(\log n)$ is required for the initial connectivity w.h.p., we have $\sqrt{\log n}$ as an asymptotic lower bound for d as well, i.e., $d = \Omega(\sqrt{\log n})$. Therefore, $d = \Theta(\sqrt{\log n})$. □

REMARK 5. In [11], the critical value of the orthogonality factor γ is proved to be $\gamma = \Theta(\frac{1}{\lambda})$ as $\lambda \to \infty$ in order to achieve the percolation for large-scale wireless networks. Recall that the degree bound is defined as $K \triangleq 1 + 1/(\gamma\beta)$ (Section 2.1) and the SINR threshold β is finite, thus $K \to \infty$ as $\gamma \to \infty$. This implies that the degree bound K should increase as n increases, and more specifically, K should grow faster than d^2; otherwise, the continuum percolation cannot be achieved as the network size goes to large.

Based on *Proposition 1*, we define d as follows.

DEFINITION 9. *Assume that $K = (1+\epsilon_n)\mu_0$, where $\epsilon_n > 0$ is an arbitrary increasing function of n, d is defined by*

$$d \triangleq c_\epsilon \sqrt{\frac{\ln n}{\lambda_0}}, \tag{18}$$

where c_ϵ is a positive constant and is chosen such that $d > r_n$ and $\exp(-\frac{2\lambda_0 d^2}{\epsilon_n^2 \mu_0}) > \frac{14}{15}$.

Note that the above definition of d guarantees $K = \omega(d^2)$ and a.a.s. E'_a given $\mu_0 = \Theta(\log n)$.

The Condition for Continuum Percolation: Now we are ready to derive the sufficient condition given in (4).

PROOF OF THEOREM 1. Based on the given conditions: $K = (1+\epsilon_n)\mu_0$ and $\mu_0 = \Theta(\log n)$, we obtain a simplified lower bound of $Pr(E'_a)$ by using (17) and (18),

$$Pr(E'_a) \geq \left(1 - \frac{1}{\epsilon_n^2 \mu_0}\right)^{2\lambda_0 d^2} \to \exp\left(-\frac{2c_\epsilon^2 \ln n}{\epsilon_n^2 \cdot \mu_0}\right), \tag{19}$$

as d is sufficiently large. Note that according to *Definition 9*, c_ϵ is chosen so that $Pr(E'_a) > \frac{14}{15}$. By substituting (15) and (19) into (12), we have the sufficient condition for continuum percolation given by

$$-\frac{6}{5}\exp\left(-\frac{\sqrt{5}}{5} c_\epsilon r_n \sqrt{\lambda_0 \ln n} S(t)\right) + \exp\left(-\frac{2c_\epsilon^2 \ln n}{\epsilon_n^2 \cdot \mu_0}\right) > \frac{14}{15}. \tag{20}$$

Let $\phi_n = 1 - \exp(-\frac{2c_\epsilon^2 \ln n}{\epsilon_n^2 \cdot \mu_0})$, then we have from (20)

$$S(t) > \frac{\sqrt{5}(\ln 18 - \ln(1 - 15\phi_n))}{c_\epsilon r_n \sqrt{\lambda_0 \ln n}}, \tag{21}$$

which is the condition given in (4). Finally, by applying *Lemma 1* and *Lemma 3*, when $S(t)$ satisfies (4), the original graph is in the super-critical phase. □

4.4 Condition for Continuum Non-percolation

In this subsection, we prove *Theorem 2* by analyzing the condition for $Pr(E_a)Pr(E'_a) < \frac{1}{9}$ because of *Lemma 3*. We first find the upper bounds for $Pr(E_a)$ and $Pr(E'_a)$, which are expected to be functions of $S(t)$, then we prove the sufficient condition for non-percolation by using these upper bounds, shown as follows.

The Upper Bound of $Pr(E_a)$: Let SLR_a^L and SLR_a^R denote the events that there is an LR-crossing in B_a^L and B_a^R, respectively, where B_a^L and B_a^R are defined in Section 4.1. Then the occurrence of event LR_a guarantees the occurrences of both events SLR_a^L and SLR_a^R, and thus

$$\begin{aligned} Pr(E_a) &= Pr(LR_a \cap TB_a^L \cap TB_a^R) \\ &\leq Pr(SLR_a^L \cap TB_a^L \cap SLR_a^R \cap TB_a^R) \\ &= Pr(SLR_a^L \cap TB_a^L) Pr(SLR_a^R \cap TB_a^R). \end{aligned} \tag{22}$$

The last equality in (22) is due to the fact that events $SLR_a^L \cap TB_a^L$ and $SLR_a^R \cap TB_a^R$ occur in disjoint sets B_a^L and B_a^R, i.e., they are independent events. We further assume that the points used for the LR-crossing in B_a^L (and

B_a^R) are different than those used for the TB-crossing in B_a^L (and B_a^R), then by *BK inequality* (*Theorem 2.3* [5]), we have

$$Pr(E_a) \le (Pr(SLR_a^L)Pr(TB_a^L))^2 = Pr(SLR_a^L)^4. \quad (23)$$

To calculate $Pr(SLR_a^L)$, we study the occurrence of the complementary event of SLR_a^L, $\overline{SLR_a} \triangleq \{$ no LR-crossing exists in B_a^L $\}$. Suppose that there is a *band* with width r_n crossing vertically through B_a^L, then the intersection of the band and B_a^L forms a rectangular with length d and width r_n, denoted by B_r. Let $SLR^c \triangleq \{$ no surviving nodes located in $B_r\}$, then $\overline{SLR_a}$ surely occurs when SLR^c occurs, which is illustrated in Figure 3. Since SLR^c is only one of causes for the occurrence of $\overline{SLR_a}$, it is obvious that $Pr(SLR^c) < Pr(\overline{SLR_a})$. Therefore, we have

$$Pr(E_a) < (1 - Pr(SLR^c))^4. \quad (24)$$

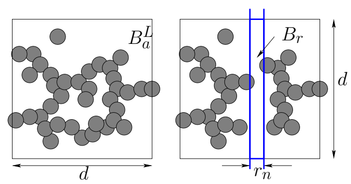


Figure 3: An illustration of the event that intercepts an LR-crossing.

As aforementioned, the point process of surviving nodes is a Poisson process with density function $\lambda_1(t) = \lambda_0 S(t)$, then we have $Pr(SLR^c) = \exp(-\lambda_0 r_n d S(t))$. With the definition of d given in (18), we have $Pr(E_a)$ upper bounded by

$$Pr(E_a) < (1 - \exp(-c_\epsilon r_n \sqrt{\lambda_0 \ln n} S(t)))^4. \quad (25)$$

Remark 6. *In (25), for any given network size n, the upper bound of $Pr(E_a)$ increases exponentially as the survival function $S(t)$ increases. Specifically, $Pr(E_a)$ goes to 0 when $S(t) \to 0$, which is in accordance with the fact that the more failed nodes in a graph, the more difficult to have a connected component in the graph.*

The Upper Bound of $Pr(E'_a)$: Recall that event E'_a happens if and only if for the edge a associated with box B_a, no point in B_a has more than K neighbors, so $Pr(E'_a) = Pr(\bigcap_{i=1}^N D_i < K)$, where N is the number of points in B_a and D_i is the degree of node i. Notice that D_i $(1 < i < N)$ are normally not independent. To obtain a reasonable upper bound of $Pr(E'_a)$, we consider the probability that all points in B_a are in a disk of radius $2r_n$, denoted by P_{2r_n}. By using the Poisson property, we have $P_{2r_n} = e^{-4\pi r_n^2}(4\pi r_n^2)^N/N!$. By the definition of d in (18), N is on the order of $\log n$. With $\mu_0 = \Theta(\log n)$, we know that P_{2r_n} goes to 0 as $n \to \infty$ ($\lim_{n\to\infty} P_{2r_n} = \lim_{n\to\infty} \frac{(\log n)^{\log n}}{n(\log n)!} = \lim_{n\to\infty} \frac{1}{n}$). This implies that w.h.p. there exist two distinct points in B_a, say u and v, such that $\|x_u - x_v\| > 2r_n$. Thus, it is reasonable to assume that in B_a there are at least two points apart from each other at a distance of at least $2r_n$. Since these two points cannot have overlapped neighbors, their node degrees are independent. Then we have $Pr(E'_a)$ bounded above by

$$Pr(E'_a) = Pr\left(\bigcap_{i=1}^N D_i < K\right) < Pr(D_i < K)^2. \quad (26)$$

Further, D_i is asymptotically Poisson distributed, we have

$$Pr(D_i < K) = \sum_{k=0}^{K-1} e^{-\mu_0}\frac{\mu_0^k}{k!} = \frac{\Gamma(K, \mu_0)}{(K-1)!}. \quad (27)$$

Then an upper bound of $Pr(E'_a)$ can be obtained as

$$Pr(E'_a) < \left(\frac{\Gamma(K, \mu_0)}{(K-1)!}\right)^2. \quad (28)$$

The Condition for Non-percolation: Now we are ready to derive the sufficient condition given in (5).

Proof of Theorem 2. By *Lemma 3*, when the open edge probability p is less than 1/9, the graph is not percolated on the discrete plane. Multiplying (25) and (28) and applying the condition above, we have

$$(1 - \exp(-c_\epsilon r_n \sqrt{\lambda_0 \ln n} S(t)))^4 \left(\frac{\Gamma(K, \mu_0)}{(K-1)!}\right)^2 < \frac{1}{9}. \quad (29)$$

Let $\psi_K = \frac{\Gamma(K,\mu_0)}{(K-1)!}$, with elementary derivations, we have

$$S(t) < \frac{\ln\sqrt{3\psi_K} - \ln(\sqrt{3\psi_K} - 1)}{c_\epsilon r_n \sqrt{\lambda_0 \ln n}}, \quad (30)$$

which is the sufficient condition given in (5). Finally, by applying *Lemma 1*, we know that when $S(t)$ satisfies (5), the original graph on the continuous plane is in the sub-critical phase, which completes the proof. □

4.5 Bounds of Critical Phase Transition Time

We have obtained the conditions under which a large-scale wireless multi-hop network is almost connected (see *Definition 2*) and fully partitioned in Section 4.3 and 4.4, respectively. In order to understand the critical phase transition time, we study two types of survival functions: light-tailed and heavy-tailed distributions.

The limits of the last connection time $t_c(n)$ and first partition time $t_p(n)$ are presented in *Corollary 1* for light-tailed survival functions and *Corollary 2* for heavy-tailed survival functions, respectively. Here we prove *Corollary 1* only and *Corollary 2* can be proved similarly.

Proof of Corollary 1. To find $t_c(n)$ under the exponential $S(t)$, substituting $S(t) = e^{-\alpha t}$ into (4), we have

$$t < \frac{1}{\alpha}\ln\left(c_\epsilon r_n \sqrt{\lambda_0 \ln n}\right) - \frac{1}{\alpha}\ln\left(\sqrt{5}\ln\frac{18}{1 - 15\phi_n}\right). \quad (31)$$

With $\lambda_0 \pi r_n^2 = \Theta(\log n)$ given in *Theorem 1*, we can assume $\lambda_0 \pi r_n^2 = c \ln n$ for some constant $c > 0$ so that the network is connected a.a.s. initially. Then we have

$$t < \frac{1}{\alpha}\ln(\ln n) + \frac{1}{\alpha}\left(\ln\left(c_\epsilon\sqrt{\frac{c}{\pi}}\right) - \ln\left(\sqrt{5}\ln\frac{18}{1 - 15\phi_n}\right)\right). \quad (32)$$

Let $c_1 = \frac{1}{\alpha}(\ln(c_\epsilon\sqrt{\frac{c}{\pi}}) - \ln(\sqrt{5}\ln\frac{18}{1-15\phi_n}))$, we have the upper limit of $t_c(n)$ given in (6) from (32).

To find the first partition time $t_p(n)$, (5) is rewritten by

$$t > \frac{1}{\alpha}\ln\left(c_\epsilon r_n \sqrt{\lambda_0 \ln n}\right) - \frac{1}{\alpha}\ln\left(\ln\frac{\sqrt{3\psi_K}}{\sqrt{3\psi_K} - 1}\right). \quad (33)$$

With $\lambda_0 \pi r_n^2 = c \ln n$ and $c_2 = \frac{1}{\alpha}(\ln(c_\epsilon\sqrt{\frac{c}{\pi}}) - \ln(\ln\frac{\sqrt{3\psi_K}}{\sqrt{3\psi_K}-1}))$, we have the lower limit of $t_p(n)$ given in (7) from (33). □

Proof of Corollary 3. It is easy to observe that $t_c(n)$ is strictly less than $t_p(n)$ according to (6)-(9). We prove $t_c(n) \le \mathcal{T}_C \le t_p(n)$ by contradiction. Assume $\mathcal{T}_C < t_c(n)$, then we can find a time point t' such that $\mathcal{T}_C < t' < t_c(n)$. By the definition of $\mathcal{T}_C$ given in (3), the network is disconnected a.a.s. at time t' (sub-critical); while by the definition of $t_c(n)$ given in (1), the network is almost connected (super-critical). This reaches a contradiction, thus $t_c(n) \le \mathcal{T}_C$. Similarly, we can prove $\mathcal{T}_C \le t_p(n)$. Therefore, $t_c(n) \le \mathcal{T}_C \le t_p(n)$, which yields the result. □

Remark 7. Although in this paper, we adopt the *extended model* (aforementioned in Section 2.1), where λ_0 is fixed and the deployment area $B(s_n)$ increases in n along with increasing r_n for $\lambda_0 \pi r_n^2 = \Theta(\log n)$, our results are applicable to the so called *constant-range model* as well [10]. In the constant-range model, r_n is fixed (independent of n) but λ_0 increases in n and $\lambda_0 \pi r_n^2 = \Theta(\log n)$ still holds. It is obvious that the deployment area $B(s_n)$ also increases in n but is at most $\Theta(\frac{n}{\log n})$. Recall that in Section 4.3, we defined the edge length d to be of the same order of transmission range r_n. If we still use this "rule of thumb" in the constant-range model, all derivations and proofs will hold, and more importantly, our final results of $t_c(n)$ and $t_p(n)$ will hold as well by considering the fact of $\lambda_0 = \Theta(\log n)$ now. Another popular way to extend a finite network to an infinite one is called the *dense model* (refer [3, 4, 9, 12] for examples), where the deployment region is usually fixed to a unit square or disk and the node density λ_0 is equal to n. Suppose $\lambda_0 \pi r_n^2 = \Theta(\log n)$ still holds, then with the same "rule of thumb", i.e., $d = \Theta(r_n)$, our results are still valid in the dense model. This generality of our results can also be explained by *Scaling theorem* in that the extended model is actually isomorphic to the dense model [5, 7, 8].

Remark 8. It is worthy of pointing out that our results can be applied to network scenarios with node mobility. Since all results are derived from the geometric random graph model with uniform node distribution, it is natural to conclude that our results are applicable when nodes are mobile, as long as the node mobility does not violate the steady-state uniform node distribution. Indeed, there have been a number of mobility models that guarantee the uniform node distribution, such as Gauss-Markov [24] and Random direction [25].

To summarize, we have addressed questions in the NPT problem formulated in Section 2 by proving the critical conditions in the network devolution due to random failures and by providing the bounds on the critical phase transition time. We evaluate our theoretical findings in the next section by simulations.

5. SIMULATION STUDY

In this section, we carry out several sets of simulations to interpret our theoretical results.

5.1 Simulation Methodology

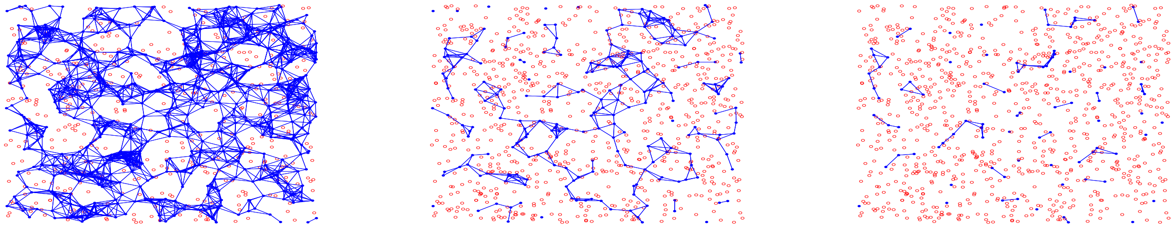
In the simulation, we distribute n nodes independently and randomly with a uniform distribution to approximate a Poisson point process. The initial node density λ_0 is set to 2.5×10^{-4} and the transmission radius r_n is carefully chosen to guarantee the full connectivity of initial topology (specifically, $\lambda_0 \pi r_n^2 \approx 5 \log n$). To implement random failures, the lifetime of each node is randomly generated according to the same distribution with parameters set below: $\alpha = 0.001$ for the light-tailed exponential survival function $S(t) = e^{-\alpha t}$ and $(\rho, \eta) = (2, 500)$ for the heavy-tailed Pareto $S(t) = (t/\eta)^{-\rho}$. To emulate the devolution process, nodes fail one by one in the increasing sequence of their lifetime. Upon node failures, we use a depth-first search (DFS) algorithm to record all components induced by surviving nodes and calculate the *giant component size* S (i.e., the number of surviving nodes in the largest component). The *relative* giant component size is defined by $S_R \triangleq S/n'$ where n' is the number of remaining surviving nodes, in order to characterize the phase transition phenomenon.

5.2 Simulation Results

Figure 4 illustrates an example of the topological devolution process of a graph of 1000 nodes, where solid dots and circles represent surviving nodes and failed nodes, respectively. The survival function is Pareto with parameters set above. By using (8) and (9) and choosing $c_\epsilon = 2.5$, we have $t_c(n) = 733.8$ and $t_p(n) = 2041$. As expected, when $t < t_c(n)$, the topology constructed by remaining nodes is almost connected with a single giant component, as shown in Figure 4(a). On the contrary, when $t > t_p(n)$, the network is fully partitioned and has only several small components, shown in Figure 4(c). Figure 4(b) shows the topology in the period of phase transition, i.e., $t_c(n) < t < t_p(n)$, where the network is disconnected into parts but with one component larger than others.

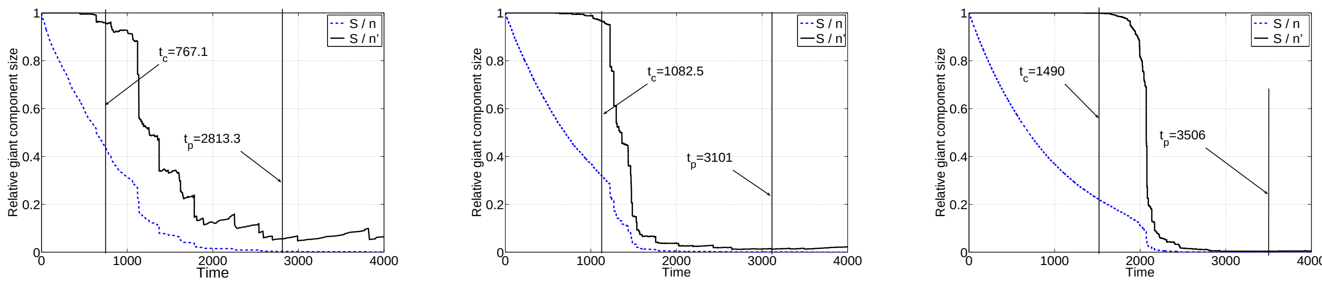
Figures 5 and 6 show clearly how the relative giant component size ($S_R \triangleq S/n'$) decreases when the network experiences increasing random failures. The analytical results of $t_c(n)$ and $t_p(n)$ are annotated on the figures, which are calculated by using (6), (7), (8), and (9) with $c_\epsilon = 2.5$. We summarize our observations as follows. First, the period of phase transition is bounded by the theoretical limits of $t_c(n)$ and $t_p(n)$ in all simulation scenarios, which confirms the correctness of our analytical results. Second, as expected, the larger the network size n is, the sharper the phase transition is, which is true for both light-tailed and heavy-tailed survival functions. Third, compared with the actual value of the giant component size S (or the ratio S/n), it is clear that the relative giant component size, i.e., $S_R = S/n'$, is a more appropriate metric to indicate the phase transition phenomenon in the devolution process due to random failures. Finally, a surprising observation is that given the same n and average node lifetime, the network with Pareto survival function decomposes substantially faster than the network with exponential survival function. To explain this phenomenon, it is noticed that the variance of Pareto-distributed lifetime is much larger than that of exponential-distributed lifetime. Actually, in the network with Pareto survival function, the majority of nodes must have short lifetimes to compensate only a few of very huge lifetimes, in order to keep the same expectation with exponential survival function. Thus, more nodes fail earlier in the network with Pareto survival functions than in the network with exponential ones.

Next, we use numerical simulations to evaluate the tightness of $t_c(n)$ and $t_p(n)$. In fact, from all figures, we can see that $t_c(n)$ is a quite tight lower bound for the period of phase transition; however, $t_p(n)$ seems to be loose. To understand the tightness of $t_p(n)$, we examine the convergence of the



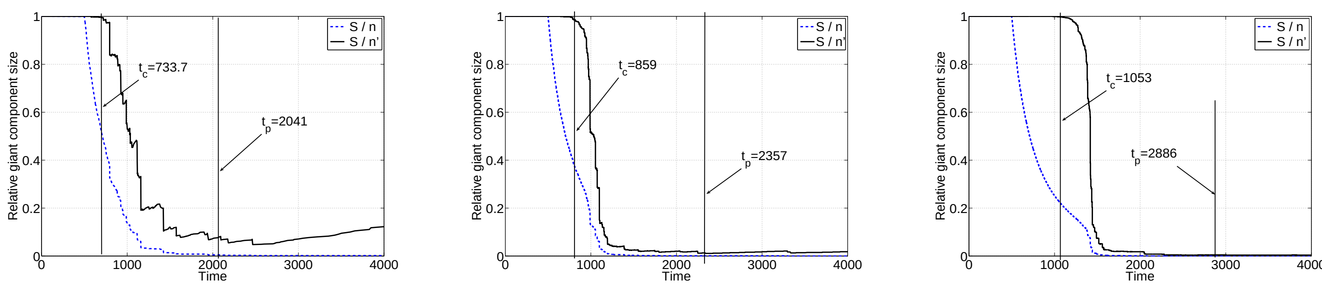
(a) $t = 729.5 < t_c(n)$ (b) $t_c(n) < t = 1204.4 < t_p(n)$ (c) $t = 2193.4 > t_p(n)$

Figure 4: Snapshots of the graph devolution process over time.



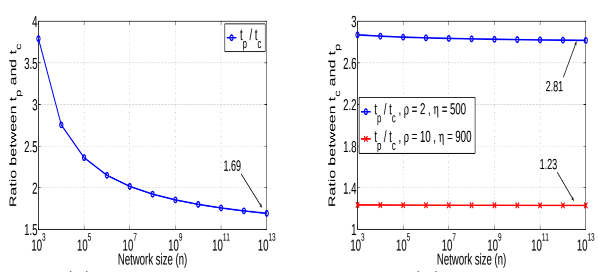
(a) $n = 1k, r_n = 138$ (b) $n = 10k, r_n = 178$ (c) $n = 10k, r_n = 195$

Figure 5: Phase transition and critical time bounds with exponential survival functions ($\alpha = 0.001$).



(a) $n = 1k, r_n = 138$ (b) $n = 10k, r_n = 178$ (c) $n = 100k, r_n = 195$

Figure 6: Phase transition and critical time bounds with Pareto survival functions ($\rho = 2.0, \eta = 500.0$).



(a) Exponential (b) Pareto

Figure 7: Tightness of t_c and t_p as $n \to \infty$ and $\rho \to \infty$.

ratio between $t_p(n)$ and $t_c(n)$ as $n \to \infty$. When $S(t)$ is light-tailed, by (6) and (7), we have $t_p(n)/t_c(n) = \frac{\ln(\ln n)/\alpha + c_2}{\ln(\ln n)/\alpha + c_1} \to 1$ as $n \to \infty$, which is illustrated in Figure 7(a). When $S(t)$ is heavy-tailed, by (8) and (9), we have $t_p(n)/t_c(n)$ equal to $c_4/c_3 \approx (\frac{\sqrt{5}\ln 18}{\ln\sqrt{3} - \ln(\sqrt{3}-1)})^{1/\rho}$. Thus, when $S(t) = (t/\eta)^{-\rho}$, the tightness of $t_p(n)$ depends on the parameter ρ only and t_p approaches $t_c(n)$ from above as ρ increasing, as shown in Figure 7(b). This observation is in accordance with the fact that if ρ is very large, all nodes may become failed at the same time since $S(t)$ approaches to the Dirac delta function $\delta(t - \eta)$ as $\rho \to \infty$. In this case, the period of phase transition goes to 0, i.e., $t_p(n)/t_c(n) \to 1$.

6. CONCLUDING REMARKS

In this paper, we studied the critical phase transition time of large-scale wireless multi-hop networks in the presence of random failures. By couping with a continuum percolation process on the geometric random graph, we obtained the conditions under which the network topology transits from an almost connected phase to a fully partitioned phase.

The lower and upper bounds of the critical phase transition time are obtained as the last connection time $t_c(n)$ and first partition time $t_p(n)$. We found that the limits of $t_c(n)$ and $t_p(n)$ are of the same order of $\log(\log n)$ and $(\log n)^{1/\rho}$, respectively, for light-tailed (exponential) and heavy-tailed (Pareto) survival functions. We finally confirmed the correctness of our theoretical analysis by simulations. An interesting result is that the network with heavy-tailed survival function is no more resilient to random failures than the network with light-tailed one, in terms of critical phase transition time, if the expected node lifetimes are identical.

It worth pointing out that the model used in this paper is rather idealized with identical circular transmission regions, uniform node distribution, and identical survival functions for all nodes. Nevertheless, we believe our results are still applicable in practical scenarios. For instance, when irregular transmission ranges have a lower bound, then transmission regions can be regarded as perfect disks with radius equal to the lower bound. With this approach, our results should provide a conservative bound on the critical phase transition time of a real network devolution process. It is interesting to evaluate our theoretical results in practice and extend our analysis with more realistic models, which will be our future work.

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8. APPENDIX

8.1 Proof of Lemma 1

Proof of Lemma 1. For any two adjacent edges (in the same direction), a and b, in $\mathcal{L}$, suppose they are associated with rectangles B_a and B_b, respectively, then B_a and B_b intersect in the same square S_{ab}, i.e., $S_{ab} = B_a \cap B_b$. If a and b are all open, then there exists at least one TB-crossing, say $\mathcal{P}_s$, in S_{ab} based on *Definition 7*. Let $\mathcal{P}_a$ and $\mathcal{P}_b$ be the LR-crossings in B_a and B_b, respectively, then both of them must intersect with the same TB-crossing in S_{ab}. This implies an

LR-crossing of the rectangle $B_a \cup B_b$, formed by $\mathcal{P}_a$, $\mathcal{P}_b$, and $\mathcal{P}_s$. If two open adjacent edges are perpendicular, we can use the similar rationale to prove that there exists a connected component crossing the two rectangles associated with the edges. In addition, based on *Definition 7*, the union of the rectangles of all edges in $\mathcal{L}$ actually covers the whole area of the graph in the continuous plane. Therefore, an infinite open edge cluster in $\mathcal{L}$ implies a giant component in $G(\mathcal{H}_{\lambda_0,s_n}, r_n)$. $\square$

Figure 8 gives an illustration for the formation of the LR-crossing described in the proof above.

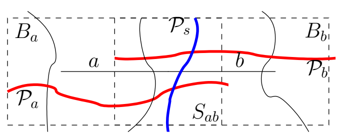

Figure 8: A long horizontal crossing formed by two adjacent open edges.

8.2 Proof of Lemma 2

Proof of Lemma 2. In the lattice $\mathcal{L}$, for any path beginning at the origin $\mathbf{0}$, it has 4 directions to choose at $\mathbf{0}$. After the first step, each new step in the path has at most 3 choices since it must avoid the previous position, and therefore $\sigma(m) \leq 4 \cdot 3^{m-1}$. Next, we estimate $\rho(m)$ in $\mathcal{L}'$ as follows. For any circuit having length m and containing $\mathbf{0}$ in its interior, it must pass through a certain vertex of the form $(kd + \frac{1}{2}d, \frac{1}{2}d)$ for $k \geq 0$. An example of such a vertex is given in Figure 1 as o'_k for $k = 1$. Note that m must be even and $m \geq 4$, thus the circuit cannot pass through $(kd + \frac{1}{2}d, \frac{1}{2}d)$ when $k \geq \frac{m-2}{2}$; otherwise the length of the circuit will be at least $m+2$. Thus, such a circuit contains a path that has a length at most $m-1$ and starts from a vertex at $(kd + \frac{1}{2}d, \frac{1}{2}d)$ for $0 \leq k\frac{m-4}{2}$. The number of such paths is at most $\frac{m-2}{2}\sigma(m-1)$, thus, $\rho(m) \leq 2 \cdot (m-2) \cdot 3^{m-2}$. $\square$

8.3 Proof of Lemma 3

Proof of Lemma 3. Let $\mathcal{C}_m$ be a circuit of the lattice $\mathcal{L}'$ with length m containing the origin in its interior, then $Pr(\mathcal{C}_m$ is closed) is equal to Pr(all m edges are closed). Based on the open edge definition described in Section 4.1, for edges a and b associated with their vicinity rectangles B_a and B_b, respectively, if a and b are not adjacent, then there is no overlap between B_a and B_b. This implies that the states of non-adjacent edges are independent. Thus at least $\lfloor \frac{m}{2} \rfloor$ edges have independent states among m edges of $\mathcal{C}_m$. Let q be the probability of any edge being closed, i.e., $q = 1 - p$, then for any $\mathcal{C}_m$, $Pr(\mathcal{C}_m$ is closed) is upper bounded by $q^{\lfloor m/2 \rfloor}$. Thus the probability that there exists a closed circuit surrounding the origin in $\mathcal{L}'$ as,

$$\sum_{\mathcal{C}_m, \forall m} Pr(\mathcal{C}_m \text{ is closed}) \leq \sum_{m=4}^{\infty} q^{\lfloor \frac{m}{2} \rfloor}\rho(m) = \frac{4(9q)^2}{9(1-9q)^2}. \tag{34}$$

Therefore, when $q < \frac{1}{15}$, i.e., $p > \frac{14}{15}$, the probability of no closed circuit surrounding the origin in $\mathcal{L}'$ is strictly greater than 0, which implies $p_\infty > 0$.

Next we prove $p_\infty = 0$ if $p < \frac{1}{9}$, which is based on fact that the largest open edge cluster is finite if and only if no infinite open path (comprised of open edges) exists. Let $\mathcal{P}_m$ be a path having a length m and beginning at the origin in $\mathcal{L}$, then with the similar logic, we can calculate the probability that there exists an open path of length m as follows:

$$Pr(\exists \text{ open path } \mathcal{P}_m) \leq p^{\lfloor \frac{m}{2} \rfloor}\sigma(m) = \frac{4}{3}(9p)^{\lfloor \frac{m}{2} \rfloor}. \tag{35}$$

If $9p$ is strictly less than 1, i.e., $p < \frac{1}{9}$, then (35) converges to 0 as $m \to \infty$, which implies no infinite open path existing in $\mathcal{L}$ and thus $p_\infty = 0$. $\square$

8.4 Proof of Lemma 7

Proof of Lemma 7. We derive the lower bound of $Pr(E_a)$ by first finding the probability that there is an LR-crossing (or TB-crossing) in a square with side length d. To facilitate the derivation, we introduce the following denotations and follow the similar proof logic of *Lemma 10.5* in [7]. Let $B(b,j) \triangleq [0, jb] \times [0, b]$ be the rectangles for $j = 1, 2$. Let SLR_b and LR_b denote the events that there is an LR-crossing in $B(b,1)$ and $B(b,2)$, respectively. Suppose $b > 0$ with $Pr(LR_b) \leq 1-\delta/25$ and $Pr(SLR_b) \leq 1-\delta/25$ for some $\delta < 1$, it is shown in [7] that for every non-negative integer m, $1-Pr(LR_{2^m b}) \leq \delta^{2^m}/25$ and $1-Pr(SLR_{2^m b}) \leq \delta^{2^m}/25$.

Let $b = \frac{\sqrt{5}}{5}r_n$, then the rectangle $B(b,2)$ is definitely within the transmission range of any surviving point that is located in $B(b,2)$. Thus, as long as there is at least one surviving point in $B(b,2)$, the event LR_b is guaranteed to occur, which enables us to derive $Pr(LR_b)$ as

$$Pr(LR_b) = 1 - \exp\left(-\frac{2}{5}\lambda_0 r_n^2 S(t)\right) \leq 1 - \frac{\delta}{25}, \tag{36}$$

where $\delta = \exp(-\frac{2}{5}\lambda_0 r_n^2 S(t))$. Then if we choose $d = 2^m b$, i.e., $2^m = \sqrt{5}d/r$, we have $Pr(SLR_d)$ bounded below by

$$Pr(SLR_d) \geq 1 - \frac{1}{25}\exp\left(-\frac{2\sqrt{5}}{5}\lambda_0 r_n d S(t)\right). \tag{37}$$

By substituting (37) into (14), (15) follows. $\square$

8.5 Proof of Lemma 8

Proof of Lemma 8. Let random variable N be the number of points in B_a, we know that for all points $0 < i < N$ events $\{D_i < K\}$ are decreasing events, where D_i is point i's degree. By using FKG's inequality, we have

$$\begin{aligned} Pr(E'_a) &= E\left[E\left[\mathbf{1}_{\{\bigcap_{i=1}^{N} D_i < K\}}|N\right]\right] \\ &\geq E\left[Pr(D_i < K)^N\right]. \end{aligned} \tag{38}$$

Since $Pr(D_i < K) \leq 1$, $Pr(D_i < K)^N$ is a convex function of N. It is easy to see that the expected number of points in B_a equals to $2\lambda_0 d^2$. By using *Jensen's inequality*, (38) is rewritten by

$$Pr(E'_a) \geq Pr(D_i < K)^{E[N]} = Pr(D_i < K)^{2\lambda_0 d^2}. \tag{39}$$

Additionally, by using *Chebyshev-Cantelli inequality* [26], we have $Pr(D_i \geq K)$ bounded by

$$Pr(D_i \geq K) = Pr(D_i - \mu_0 \geq k\sqrt{\mu_0}) \leq \frac{1}{1+k^2} \tag{40}$$

Combining (39) with (40), (17) follows. $\square$