SPECTRAL ESTIMATION IN HIGHLY TRANSIENT DATA

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ABSTRACT

We propose a new framework for estimating different frequencies in piece-wise periodic signals with time varying amplitude and phase. Through a 3-dimensional delay embedding of the introduced model, we construct a union of intersecting planes where each plane corresponds to one frequency. The equations of each of these planes only depend on the associated frequency, and are used to calculate the tone in each segment. A sparse subspace clustering technique is utilized to find the segmentation of the data, and the points in each cluster are used to compute the normal vectors. In the presence of white Gaussian noise, principal component analysis is used to robustly perform this computation. Experimental results demonstrate the effectiveness of the proposed framework.

Index Terms— Spectral estimation, transient data, delay embedding, sparse subspace clustering

1. INTRODUCTION

Spectral estimation using sampled data of sinusoidal signals is a well studied problem in signal processing with applications in many areas such as biomedical signal processing, speech processing and communications. A variety of approaches to this problem has been developed, and many are explicitly based on an additive model of sinusoids with constant amplitude and phase for each tone embedded in additive white noise [1–3]. Although this model has been an effective way to represent a variety of signals, many waveforms with almost periodic structures fail this formulation because of the inherent non-stationarity. In this study, we use a piecewise sinusoidal function where each tone has time varying amplitude and phase. This representation is an effective tool for expressing the periodic structures in biomedical signals such as wheezes and cell cycle regulated genes [4–6], speech processing [7], image compression artifacts [8] and patterned texture analysis [9].

Accurate and robust estimation of the frequencies in the introduced model is remarkably beneficial in practice.

Enumerating and estimating the frequencies in wheezes is a premise to essential diagnostic knowledge and can be very useful in visualization of information in health monitoring devices. The number of tones present in a recorded wheeze signal is an indicator of how many airway occlusions have occurred in the respiratory tree. Wheezes with higher frequencies, usually referred to as high pitched wheezes, are associated with obstruction of the small airways while low pitched wheezes with lower frequencies are related to diseases of larger airways. In addition, the frequency of a microarray time series profile from a periodically expressed gene provides important information about cell cycle regulation. Particularly, if this frequency is the same as the cell cycle, we will be able to identify cell cycle regulated genes. Thus, we propose a framework for frequency estimation in the group of signals with almost harmonic patterns that can be well represented by the introduced model.

The delay coordinate embedding method which embeds a time series into a higher dimensional space, was first proposed by Takens [10] with the goal of recovering the underlying dynamics of a system using its output. This technique has been mostly used in the analysis of dynamical systems and chaotic attractors [11–13]. Two dimensional delay embeddings of time series were also employed in signal analysis in [4–6] to detect almost harmonic patterns. Moreover three dimensional delay embeddings have been used in human speech recognition in [14]. In this study, we propose a three dimensional delay embedding for frequency estimation and prove that each frequency in the time domain will correspond to a plane in embedding space. To that end, we exploited a recently proposed algorithm based on sparse representation methods, called Sparse Subspace Clustering (SSC) [15] to cluster the whole set of points into 2 dimensional subspaces (planes). The points on each subspace can be used to calculate the equation of the plane which is completely determined by the frequency. In the noise free case, only three points in each subspace would be sufficient for this computation. However, in the presence of noise, principal component analysis (PCA) is employed for estimating the plane normal vector . The proposed algorithm is robust to missing data points, time varying sampling rates and noise. Moreover, since only a few sub-samples are sufficient for a normal vector computation,

it performs well for highly downsampled signals. Real time frequency estimation with very low computational cost can hence be achieved using the proposed approach.

The remainder of the paper is organized as follows: the signal model, the time delay embeddings and their relations to underlying frequencies in a sparse subspace clustering framework are discussed in Section 2. Experimental results are included in Section 3. Finally Section 4 concludes the paper.

2. PROPOSED FRAMEWORK

2.1. Signal Model

The proposed signal model is a continuous piecewise sinusoidal function with different periods and phase with a time varying envelope defined as

$$y(t) = \sum_{i=1}^{n} g_i(t), \qquad (1)$$

where

$$g_i(t) = \begin{cases} y_i(t) & t_{i-1} \le t < t_i, \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

and $y_i's, i = 1, 2, ..., n$ are defined as,

$$y_i(t) = A(t) \sin(\omega_i t + \phi_i), \qquad (3)$$

where $A(t)$ is a nonzero continuous amplitude function and $\phi_{i+1} = \phi_i + t_i(\omega_i - \omega_{i+1})$ satisfies the continuity of $y(t)$.

This representation has been used for periodicity detection in breathing sound signals with the goal of wheeze detection, since the harmonic pattern of wheezes in the time domain are well represented by this formulation [4, 5]. It has also been utilized to identify the cell cycle regulated genes in [6], as it suitably fits the periodic patterns of cell cycles in genetic expression datasets.

2.2. 3D Time Delay Embeddings

In a general representation of delay coordinate embedding, for each time series with the sampling time T_s denoted by $y(t), t = \{i.T_s\}, i = 1, 2, ...$, the following vector quantity of m components is constructed:

$$Y_m(t) = (y(t), y(t + \tau_1), y(t + \tau_2), ..., y(t + \tau_{m-1})), \quad (4)$$

where $y(t) \in \mathbb{R}$, $\tau_1, ..., \tau_m$ are different time delays, and m is the embedding dimension. The delay times and the embedding dimension are essential parameters to determine. The appropriate interval for choosing a delay time to best obtain informative delay embedding of signal is $t_{c1} < \tau < t_{c2}$, where t_{c1} and t_{c2} are the first and second critical points of an autocorrelation-like function defined as

$$R_{yy}(t) = \sum_k y(k + t)y(k). \qquad (5)$$

Equation (4) with $m = 2$ is used in [4] to detect almost periodic patterns. The 2-dimensional time delay embedding $Y_2(t)$ is a set of concentric ellipses with major axis angles of rotation $\pm\pi/4$ and varying radii due to different frequencies. The varying side lengths of the circumscribed squares around these ellipses are due to a time-varying amplitude [4].

In this study, we use $m = 3$ for estimating different frequencies present in the signal. The following Theorem shows that three dimensional time delay embedding of a sinusoidal signal will form a plane whose equation only depends on the frequency.

Theorem 1. *Suppose that* $\tau_1, \tau_2 \ne \frac{k\pi}{\omega}$. *The 3 dimensional delay-coordinate embedding of* $s(t) = \sin(\omega t)$ *obtained using (4) as* $S(t) = (\sin \omega t, \sin \omega(t + \tau_1), \sin \omega(t + \tau_2))$ *lies on a plane with the normal vector*

$$N = (\sin \omega(\tau_1 - \tau_2), \sin \omega\tau_2, -\sin \omega\tau_1). \qquad (6)$$

Proof. We can write the following equations for the last two terms of $S(t)$ using trigonometric properties

$$\sin(\omega t + \omega\tau_i) = \sin(\omega t)\cos(\omega\tau_i) + \sin(\omega\tau_i)\cos(\omega t) \qquad (7)$$

where $i = 1, 2$. Equating the term $\cos \omega t$ in the equations for $i = 1$ and $i = 2$ yields $\operatorname{cosec}(\omega\tau_1)\sin(\omega t + \omega\tau_1) - \cot(\omega\tau_1)\sin(\omega t) - \operatorname{cosec}(\omega\tau_2)\sin(\omega t + \omega\tau_2) + \cot(\omega\tau_2)\sin(\omega t) = 0$. By rearranging the terms in the above equation, we obtain the following

$$(\cot(\omega\tau_2) - \cot(\omega\tau_1))\sin(\omega t) + \operatorname{cosec}(\omega\tau_1)\sin(\omega t + \omega\tau_1)$$
$$-\operatorname{cosec}(\omega\tau_2)\sin(\omega t + \omega\tau_2) = 0. \qquad (8)$$

The equation above is a plane equation where $\sin(\omega t)$, $\sin(\omega t + \omega\tau_1)$ and $\sin(\omega t + \omega\tau_2)$ are the three coordinates. Thus, by multiplying sides by $\sin(\omega\tau_1)\sin(\omega\tau_2)$ we can obtain the normal vector as expressed in Equation (6). $\square$

Time delay embedding of $\sin(\omega t + \phi_1)$ and $\sin(\omega + \phi_2)$ only differ by a reparametrization and are therefore eventually equal sets [4]. Accordingly, Theorem 1 is also valid when a phase term is added. Moreover, since both sides of Equation (7) can be multiplied and divided by nonzero continuous function $A(t)$, the theorem is valid for $y_i(t)$ as expressed in (3). Thus, we can conclude the following,

Suppose that the time delays τ_1 *and* τ_1 *are selected between* t_{c1} *and* t_{c2}. *The 3-dimensional delay embedding* $Y_3(t)$ *of* $y(t)$ *as described in Equations (1)-(3) forms a set of* n *intersecting planes where the points lying on the* i^{th} *plane correspond to the frequency* ω_i *in the time domain signal.*

The extension of this statement to delay embeddings of any dimension higher than 3 can clearly follow. In order to estimate n frequencies in the signal expressed by Equations (1)-(3), we need to find the clusters of data points which lie

on each plane. Sparse Subspace clustering technique introduced by [15] as described in the next subsection, is used for accomplishing this.

After clustering, we can calculate the normal vector of each plane using three randomly selected points from each segment. In the presence of additive white Gaussian noise, PCA is however employed for this computation. The frequencies then can be obtained by comparing the calculated vector with Equation (6).

2.3. Sparse Subsapce Clustering

Sparse subspace clustering (SSC) is an algorithm for clustering a set of lower dimensional subspaces in a higher dimensional space using sparse representation methods. In this study, we use SSC to cluster points lying on a collection of two dimensional subspaces or planes. The problem is establishing the membership of each data point to each particular subspace, as well as calculating the basis of the subspaces.

Consider $\{S_l\}_{l=1}^n$ as a collection of n linear subspaces of dimensions $\{d_l\}_{l=1}^n$ in a D-dimensional space. $\{y_i\}_{i=1}^N$ represents N data points lying in the union of n subspaces. The matrix $\mathbf{Y}$ including all the points in the dataset, can be written as

$$\mathbf{Y} = [y_1 \dots y_N] = [\mathbf{Y}_1 \dots \mathbf{Y}_n]\Gamma, \qquad (9)$$

where $\mathbf{Y}_l$ is a $D \times N_l$ matrix including N_l data points lying on the subspace S_l, and Γ denotes a permutation matrix that is not known a priori. This clustering technique is based on the self-expressiveness characteristic of the data described as follows. Each point y_i in a union of subspaces $\bigoplus_{l=1}^n S_l$ can be represented using a combination of other points in the dataset as $y_i = \mathbf{Y} \boldsymbol{c_i}$, where $\boldsymbol{c_i} = [c_{i1} \ c_{i2} \dots c_{iN}]^T$. Note that $c_{ii} \neq 0$ to make sure a point is written as a linear combination of other points excluding itself. Generally, the representation of y_i in this format in not unique. There is however a unique solution called subspace-sparse representation where the nonzero elements of $\boldsymbol{c_i}$ correspond to the points that belong to the same subspace as the point y_i. In the ideal case of sparse representation, the point y_i in the subspace S_l can be written as a linear combination of points lying on the same subspace. This solution can be obtained by solving the following optimization problem
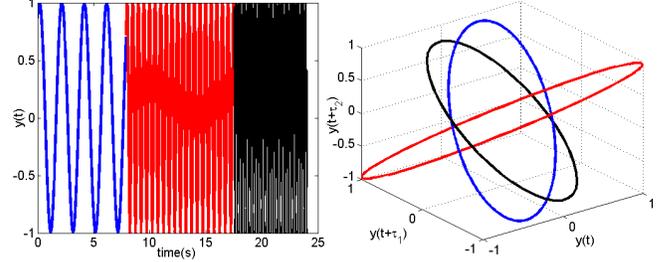
$$\min \|\boldsymbol{c_i}\|_1 \quad s.t. \quad y_i = \mathbf{Y}\boldsymbol{c_i}, \ c_{ii} = 0, \qquad (10)$$

which can be rewritten in matrix form as

$$\min \|\mathbf{C}\|_1 \quad s.t. \quad \mathbf{Y} = \mathbf{Y}\mathbf{C}, \ \text{diag}(\mathbf{C}) = 0, \qquad (11)$$

where $\mathbf{C}$ is a block diagonal matrix including the sparse representations of y_i's, i.e. $\mathbf{C} = [\boldsymbol{c_1} \ \boldsymbol{c_2} \dots \boldsymbol{c_N}]$ and $\text{diag}(\mathbf{C})$ denotes diagonal elements of $\mathbf{C}$.

In order to cluster the data after solving the optimization problem, we need to construct a weighted graph with N data

Fig. 1: left: y(t) with constant amplitude and 3 different frequencies in different colors. Right: delay embedding $Y_3(t)$

(a)   (b)
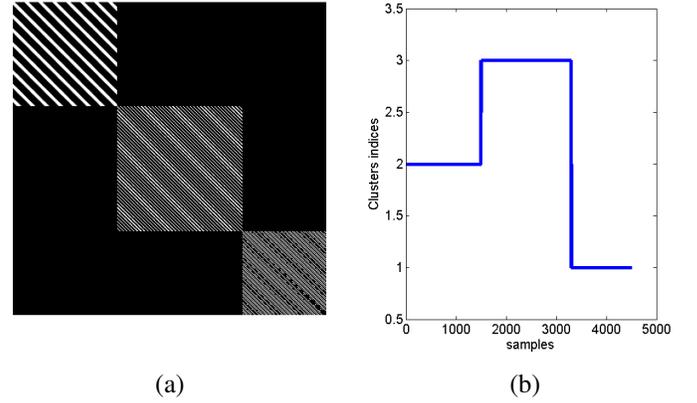
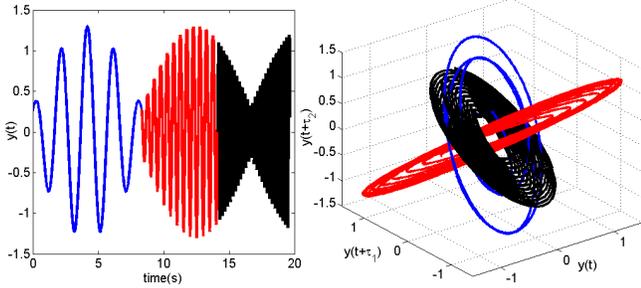**Fig. 2**: (a) matrix of sparse coefficients, (b) clusters indices

points as its nodes. The symmetric similarity matrix of the edges weights is denoted by $\mathbf{W} \in \mathbb{R}^{N \times N}$. In the ideal case, there will be edges only between the nodes corresponding to the points lying on the same subspace. Accordingly, the sparse coefficients matrix $\mathbf{W} = |\mathbf{C}| + |\mathbf{C}|^T$ expressed below is a suitable choice.

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{W}_n \end{bmatrix} \Gamma \qquad (12)$$
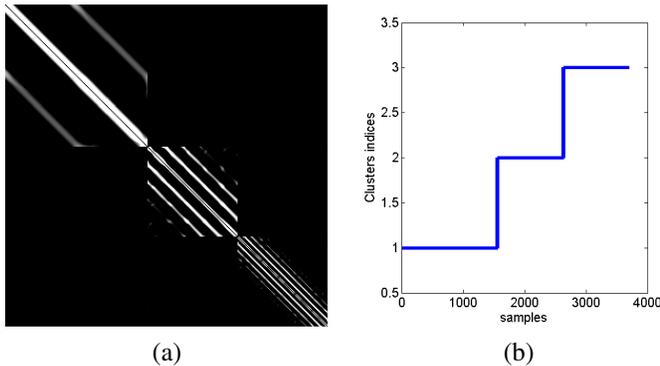
where $\mathbf{W}_l$ represents the similarity matrix of the points lying on subspace l. Finally, the spectral clustering [16] of the described graph will yield segmentation of the data points into lower dimensional subspaces. In other words, we normalize and stack the n largest eigenvectors of the Laplacian matrix of the graph in the columns of a matrix and apply the K-means method [17].

3. EXPERIMENTAL RESULTS

Numerical simulations have been performed to evaluate the performance of the proposed frequency estimation approach for synthesized piecewise sinusoidal signals with different frequencies, and time varying amplitude embedded in additive white Gaussian noise. To asses the robustness of the
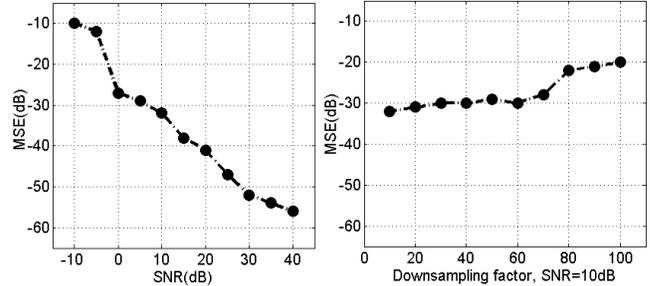
Fig. 3: left: y(t) with time varying amplitude and 3 different frequencies. Right: delay embedding

(a)   (b)

Fig. 4: (a) matrix of sparse coefficients, (b) clusters indices

Fig. 5: left: Mean square frequency estimation error versus SNR, right: MSE versus downsampling factor for SNR = 10dB

introduced technique to noise and downsampling, we investigate its accuracy for different signal to noise ratios (SNR) and different downsampling factors. The Mean Square Error (MSE) denoted by $E\{\sum_{i=1}^{n}(\hat{\omega}_i - \omega_i)^2\}$, is employed as the performance measure.

In the first test, we consider frequency estimation for a signal represented by Equations (1) - (3) with sampling frequency of 200, three different frequencies and constant amplitude. The sinusoidal parameters are $f_i \in \{0.5, 2, 5\}$ and $A(t) = 1$. Figure 1 shows the described signal with different frequencies represented in different colors and their corresponding three dimensional delay embedding. The second zero of an autocorrelation-like function of the signal was chosen as the delay for the delay embedding construction. Clearly, the delay embedding forms three ellipses corresponding to the three tones in time domain, each lying on a different plane. Figure 2 illustrates the results of the sparse subspace clustering performed on time delay embedding of the signal shown in Figure 1. Note that in order for the SSC algorithm to perform accurately, the dimension of the ambient space D needs to be greater than the dimensions of the subspace $\{d_l\}_{l=1}^{n}$. Since this study deals with the planes, i.e. $d_1 = d_2 = d_3 = 2$, we have chosen an arbitrary large value as the embedding dimension ($D = 20$). The sparse coefficients matrix $\mathbf{W}$ shown in Figure 2 - (a) includes three blocks corresponding to the three lower dimensional subspaces. We

have assigned one index from the set $\{1, 2, 3\}$ to each cluster. The membership of each point in the delay embedding to each of these clusters is represented in Figure 2- (b) validating the correctness of the clustering method.

In the second experiment, we consider a signal with similar characteristics but with time varying amplitude $A(t)$ as shown in Figure 3. In this case, the 3-dimensional delay embedding of each tone is a set of ellipses with different sizes of the circumscribed squares while they are still lying on the same plane (Figure 3). Figure 4 represents the clustering results including the sparse coefficients matrix and the clustering groups indices.

White Gaussian noise with different SNRs has been added to the signal, and PCA is used for finding the normal vector of each plane after clustering. Equation (6) is then employed to estimate the frequencies. Figure 5 shows the MSE of the frequency estimation versus SNR when $N = 50$ trials are carried out for each SNR. The validation results for the robustness of the proposed technique to downsampling is represented in the plot on the right of Figure 5, where the MSE of the frequency estimation is shown for different downsampling factors while SNR is fixed at 10 dB. The original sampling frequency of 2kHz is used for the last plot.

4. CONCLUSION

In this study, we proposed higher dimensional delay coordinate embedding as a tool to estimate distinct frequencies in almost harmonic signals modeled as piecewise sinusoids, where each tone has a time varying amplitude and phase. The delay embedding point cloud forms a union of planes where each plane corresponds to one frequency in the time domain. Sparse subspace clustering is used to segment lower dimensional subspaces in this point cloud, and a normal vector of each segment is obtained using PCA in presence of additive white Gaussian noise. The introduced technique is robust to high downsampling factors, missing data points and uneven sampling rates. The numerical results validate the promising performance of the proposed algorithm.

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